ARTICLE IN PRESS

OPEN ACCESS

Journal of Multidisciplinary Applied Natural Science Vol. xx No. xx (xxxx) https://doi.org/10.47352/jmans.2774-3047.430

Check for updates

Research Article

Modeling Lifetime Data using the Hybridization Rama Distribution: Construction, Properties, Bayesian Estimation with Real-World Applications

Waleed M. Afify, Eslam Abdelhakim Seyam*, Ahmed E. Khairalla, Emadeldin I. A. Ali, Said G. Nassr, and Ahmed Ramzy Shehata

Received : September 10, 2025 | Revised : February 22, 2026 | Accepted : March 30, 2026 | Online : May 13, 2026

Abstract

This article introduces a novel hybrid probability distribution, termed the hybridization Rama (Hyb-R) distribution, constructed by integrating the proportional hazard model (PHM) within the rank transmutation map (RTM) framework, using the Rama distribution as a baseline. The proposed model aims to enhance flexibility and improve the modeling of lifetime data with skewness and kurtosis properties that standard distributions often fail to capture. We derive and explore the structural properties of the Hyb-R distribution, including its density and distribution functions, survival characteristics, and statistical measures. Parameter estimation is conducted using both classical methods via maximum likelihood estimation (MLE) and Bayesian methods under various loss functions (SEL, LINEX, and GEL), with implementation via the Metropolis-Hastings (MH) algorithm of Markov Chain Monte Carlo (MCMC) methods. A simulation study is performed to assess the accuracy and efficiency of the estimators. Results reveal that Bayesian estimators, particularly under informative priors and GEL, outperform MLEs in terms of lower mean square error and shorter credible intervals. The proposed Hyb-R model provides a flexible and effective alternative for modeling censored and real-world lifetime data.

Keywords: bayesian estimation, hybridization rama distribution, Markov chain Monte Carlo, maximum likelihood estimation, Metropolis-hastings algorithm, proportional hazard model, rank transmutation map

1. INTRODUCTION

Extending an established distribution can significantly enhance its behavioral range and provide a more flexible family of distributions, allowing for the modeling of various types of data. Therefore, it is essential to encourage researchers to explore and develop new, more adaptable distributions by introducing additional parameters to existing ones. Gupta and Kundu [1]-[3] discussed different methods, for example, the proportional hazard method (PHM), which can be used in introducing additional shape/skewness parameters to a base probability model. The PHM model is regarded as one of the most popular models in survival analysis, [4]. Nadarajah and Kotz [5]-[7] introduced a distribution which generalizes the standard Fréchet distribution and, in the same way, the generalization of the standard exponential distribution, and studied its mathematical properties. Also, they introduced a distribution that generalizes the standard Gumbel distribution. Srinivasa-Rao et al. [8] introduced the type II exponentiated log logistic distribution by defining a new distribution by the survival function.

The concept of reversed hazard rate has attracted considerable interest from researchers in survival analysis and reliability theory. Mudholkar and Srivastava [9] introduced the exponentiated Weibull family. Shirke and Kakade [10] and [11] have analyzed the sets of real-life data and stated that the exponentiated lognormal distribution is better than that of the Weibull and the exponentiated exponential distribution. Thus, the exponentiated lognormal distribution can be regarded as an effective alternative. Sarhan and Kundu [12] introduced a new distribution called the generalized linear failure rate distribution. They discussed the properties of this distribution. Also, Gupta and Kundu [13] defined the proportional reversed hazard logistic distribution family as the proportional reversed hazard family with the baseline distribution as the logistic distribution. There are many statistical distributions for modeling datasets occurring in applied sciences, finance, engineering, and insurance, among others. A certain statistical distribution may be useful for a

Publisher's Note:
Pandawa Institute stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Please cite this article as: W. M. Afify, E. A. Seyam, A. E. Khairalla, E. I. A. Ali, S. G. Nassr, and A. R. Shehata. (2026). "Modeling Lifetime Data using the Hybridization Rama Distribution: Construction, Properties, Bayesian Estimation with Real-World Applications". *Journal of Multidisciplinary Applied Natural Science*. https://doi.org/10.47352/jmans.2774-3047.430

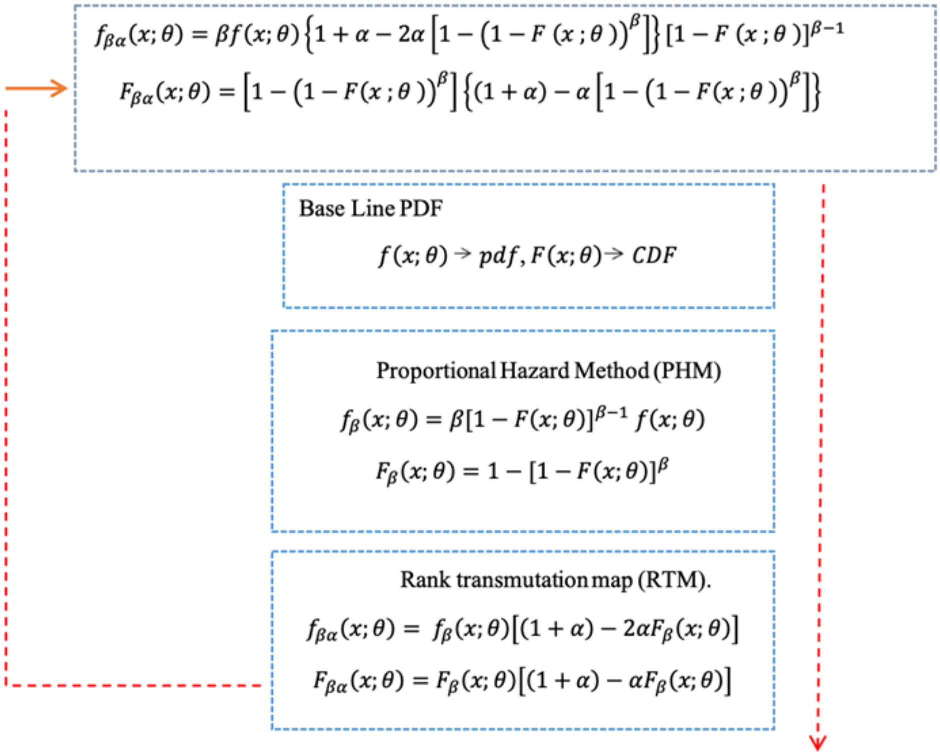


Figure 1. PDF and CDF using hybridization PHM in RTM.

particular dataset, but for a different dataset, it may not be useful due to the features of the data being analyzed. In view of this challenge, researchers have, over the years, channeled efforts towards developing methods of generating new probability distributions to enhance their capability to fit datasets that have a high degree of skewness and kurtosis. Such extended distributions have been found to provide greater flexibility in modelling several kinds of datasets, see, Hassan et al. [14]-[15], Hassan and Nassr [16], Hassan et al. [17], Abushal et al. [18], Hassan and Nassr [19], Ahmadini et al. [20], Sharma et al. [21], Nassr et al. [22], Gairola et al. [23], Younus et al. [24], Kamal et al. [25]-[26], El-Saeed et al. [27], Al Mutairi et al. [28], Tang et al [29], Mahmoud et al. [30], Atchadé et al [31], Ahmed et al. [32] and Mahmoud et al. [33].

Even though there are many different versions of the Rama distribution, most models are still built using just one transformation method. This makes it harder for them to predict different types of hazard rate behaviours and tail features. Furthermore, the integration of the PHM and rank transmutation map (RTM) within a cohesive hybridisation framework, alongside an extensive Bayesian estimation analysis utilising various loss functions and real-world applications, remains insufficiently explored in existing literature. This gap drives the creation of the suggested hybridizationRama distribution.

Recently, a new distribution called the Rama distribution was proposed and studied by Shanker [34]. Suppose x follows a Rama distribution with parameter θ. Then, the probability density function (PDF) and the cumulative density function (CDF) of the Rama distribution, respectively, are given by Equations (1) and (2).

$$f(x;\theta) = \frac{\theta^4}{\theta^3+6}(1+x^3)e^{-\theta x} \; ; \quad x,\theta > 0 \tag{1}$$

$$F(x;\theta) = 1-\left[1+\frac{\theta^3x^3+3\theta^2x^2+6\theta x}{\theta^3+6}\right]e^{-\theta x} \; ; \quad x,\theta > 0 \tag{2}$$

However, the Rama distribution contains only a scale parameter θ and, for this reason, is not flexible for statistical modeling. To improve the flexibility of the Rama distribution in modeling datasets with a variety of shapes, some extensions of the Rama distribution have been given, such as found in Vijayakumar et al. [35]-[36], Abeb et al. [37], Chrisogonus et al. [38], Alzaatreh et al. [39], Aryal and Tsokos [40], Ayele and Alemu [41], Abdel

Hady [42], Afify [43], Onyekwere et al. [44] and Srinivasa-Rao et al. [45]. This paper aims to introduce a new technique called hybridization distribution based on rank transmutation maps. We can use the same method in various methods based on Rama distribution.

2. HYBRIDIZATION PROCESS AND PHM IN RTM (HYB–R DISTIBUTION)

To get a more flexible family of distributions for modeling various types of data, we will add parameters to a well-established distribution that can be greatly effective in expanding the behavior range of this distribution. The hybridization process is a process of substituting one of the methods that adds new parameters to the family of distributions in other methods. In this paper, we will concentrate on the PHM and RTM. We note that the arrangement is important; it means that substituting PHM in RTM will produce a different distribution from substituting RTM in PHM. It is assumed that base line has Rama distribution with known parameters, where the proposed distribution provides several standard distributions as special cases.

Using the baseline PDF of the Rama distribution, the new PDF and CDF are derived by applying the hybridization process (RHM) under the RTM, incorporating the scale parameters θ, β, and the shape parameter α. These functions are graphically illustrated in Figure 1;

$$f_{\beta\alpha}(x;\theta) = \frac{\beta\theta^4 e^{-\theta x}}{u_{2(\theta)}}(1+x^3)\left\{1+\alpha-2\alpha\left[1-\left(\left[1+w_{(\theta)}\right]e^{-\theta x}\right)^{\beta}\right]\right\}\left[\left[1+w_{(\theta)}\right]e^{-\theta x}\right]^{\beta-1}; x \geq 0, \theta > 0, \beta > 0, 1 > \alpha > 0 \quad (3)$$

$$F_{\beta\alpha}(x;\theta) = \left[1-\left(\left[1+w_{(\theta)}\right]e^{-\theta x}\right)^{\beta}\right].\left\{(1+\alpha)-\alpha\left[1-\left(\left[1+w_{(\theta)}\right]e^{-\theta x}\right)^{\beta}\right]\right\}. \quad (4)$$

where, $u_{1(\theta)} = \theta^3 x^3 + 3\theta^2 x^2 + 6\theta x$, $u_{2(\theta)} = \theta^3 + 6$ and $w_{(\theta)} = \frac{u_{1(\theta)}}{u_{2(\theta)}}$. $f_{\beta\alpha}(x;\theta)$ in Equation (3) is a very

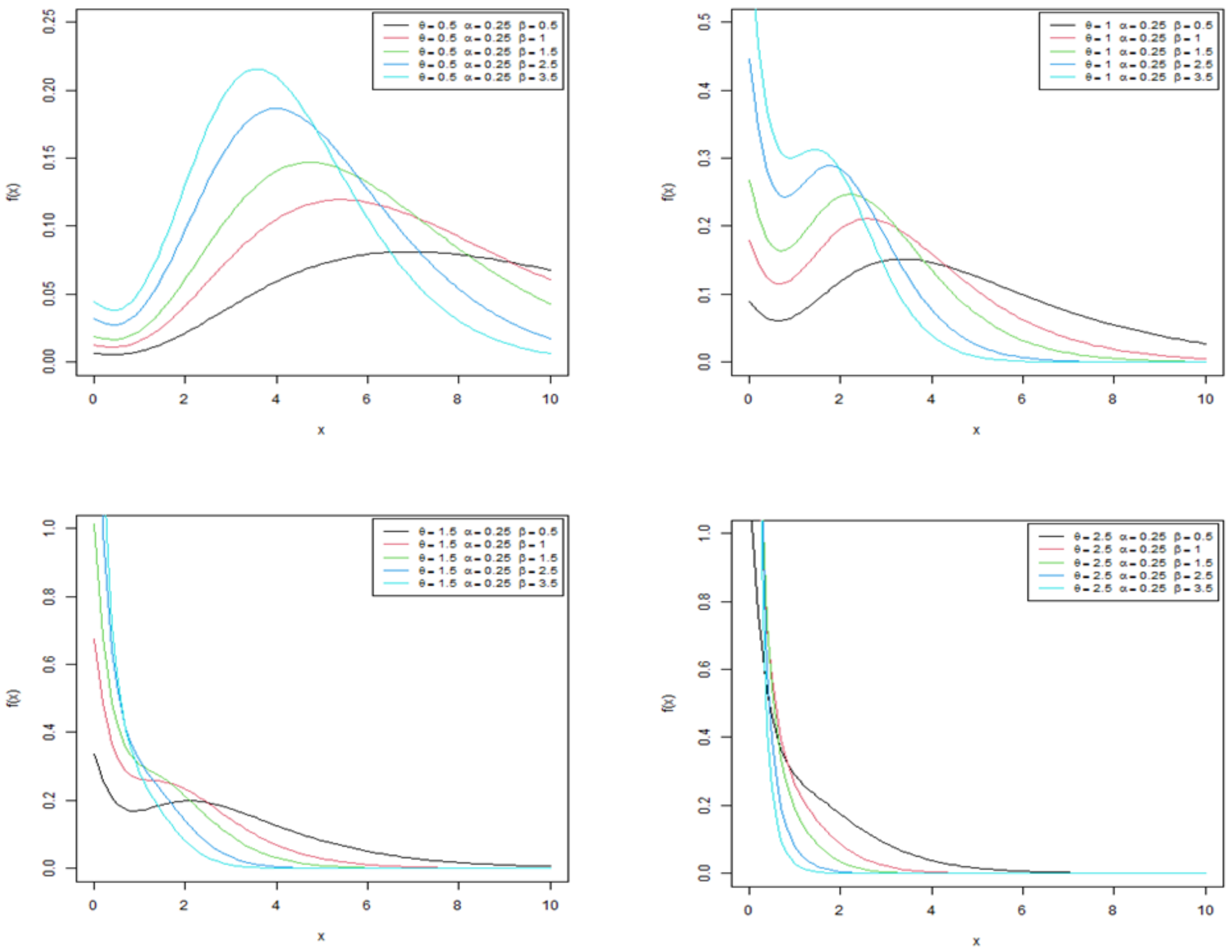


Figure 2. Behavior of the PDF for varying values of parameter θ, α, and β using Hyp-R distribution.

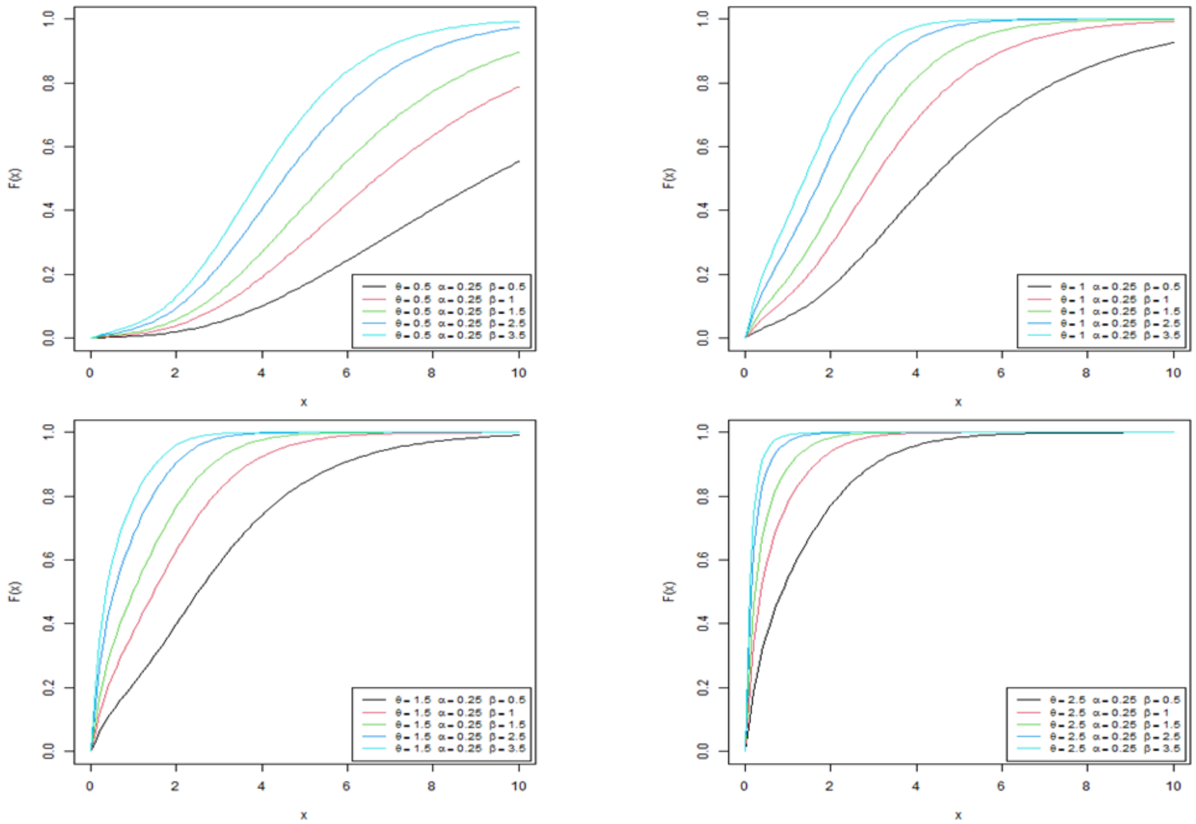


Figure 3. Behavior of the CDF for varying values of parameter θ, α, and β using Hyp-R distribution.

elastic distribution when its parameters are changed. When α = 0, $f_{-\beta 0}(x;\theta) = f_{\beta}(x;\theta)$ and if $\alpha = 0$, $\beta = 1$, and $f_{10}(x;\theta) = f(x;\theta)$.

As illustrated in Figure 1, the hybridization process incorporates the PHM within the RTM framework. This construction reflects an innovative statistical engineering strategy that transforms the baseline distribution via PHM and then maps it through RTM to preserve the underlying data ranking. The result is a distribution that adapts well to varying data complexities and provides more modeling accuracy than traditional approaches.

Various graphs of the of Hyb-R distribution for varying values of parameters have been drawn and presented in Figure 2. As the value of the shape parameter alpha increases, the graph of Hyb-R distribution approaches the normal distribution. Various graphs of the CDF of Hyb-R distribution for varying values of its parameters have been drawn and shown in Figure 3.

3. RELIABILITY ANALYSIS

We present the survival function, the hazard rate function, the reversed hazard rate function, the cumulative hazard rate function, and the mean residual lifetime for Hyb-R distribution. The survival function (SF) of Hyb-R distribution is denoted by $S(x)$, is given by Equation (5).

$$S(x) = 1 - \left[1 - \left(\left[1 + w_{(\theta)}\right]e^{-\theta x}\right)^{\beta}\right].\left\{(1+\alpha) - \alpha\left[1 - \left(\left[1 + w_{(\theta)}\right]e^{-\theta x}\right)^{\beta}\right]\right\} \quad (5)$$

The other characteristic of interest of a random variable is the hazard rate function (HRF) $h(x)$. The HRF of Hyb-R distribution is given by Equation (6).

$$h(x) == \frac{\frac{\beta\theta^4}{u_{2(\theta)}}(1+x^3)e^{-\theta x}\left\{1+\alpha-2\alpha\left[1-\left(\left[1+w_{(\theta)}\right]e^{-\theta x}\right)^{\beta}\right]\right\}.\left[\left[1+w_{(\theta)}\right]e^{-\theta x}\right]^{\beta-1}}{1-\left[1-\left(\left[1+w_{(\theta)}\right]e^{-\theta x}\right)^{\beta}\right]\left\{(1+\alpha)-\alpha\left[1-\left(\left[1+w_{(\theta)}\right]e^{-\theta x}\right)^{\beta}\right]\right\}} \quad (6)$$

We note that $h(x)$ might be increasing, decreasing, or bathtub shaped depending on the values of the parameters involved.

The reversed hazard rate function, denoted by $r(x)$, of a random variable distributed according to the Hybridization Rama distribution is given by Equation (7).

$$r(x) = \frac{\frac{\beta\theta^4}{u_{2(\theta)}}(1+x^3)e^{-\theta x}\left\{1+\alpha-2\alpha\left[1-\left(\left[1+w_{(\theta)}\right]e^{-\theta x}\right)^{\beta}\right]\right\}.\left[\left[1+w_{(\theta)}\right]e^{-\theta x}\right]^{\beta-1}}{\left[1-\left(\left[1+w_{(\theta)}\right]e^{-\theta x}\right)^{\beta}\right]\left\{(1+\alpha)-\alpha\left[1-\left(\left[1+w_{(\theta)}\right]e^{-\theta x}\right)^{\beta}\right]\right\}} \quad (7)$$

The cumulative hazard rate function of Hyb-R distribution is given by Equation (8).

$$H(x) = \int_0^x h(t)\,d\,t = -ln\,S(x)$$
$$= -ln\left\{1 - \left[1 - \left(\left[1 + w_{(\theta)}\right]e^{-\theta x}\right)^{\beta}\right].\left\{(1+\alpha) - \alpha\left[1 - \left(\left[1 + w_{(\theta)}\right]e^{-\theta x}\right)^{\beta}\right]\right\}\right\} \quad \textbf{(8)}$$

The behaviors of hazard rate function of Hyb-R distribution for varying values of parameters have been shown graphically in Figure 4. The graphs of the hazard rate function of Hyb-R distribution are monotonically increasing for varying values of parameters.

4. STATISTICAL PROPERTIES

The statistical properties of the Hyb-R distribution are analyzed, including the moments, the moment generating function, numerical results, the quantiles, and the median. The q_{th} quantile function (x_q) Hyb-R distribution is given by Equation (9).

$$x_q = F_{\beta\alpha}\left(x_q;\theta\right)^{-1}, 0 < q < 1 \quad \textbf{(9)}$$

This method is effective for generating a Hyb-R random variable. Furthermore, the median (m) of X is $q = 0.5$ can be obtained numerically.

The r^{th} non-central moment of Hyb-R distribution is given by Equations (10) and (11).

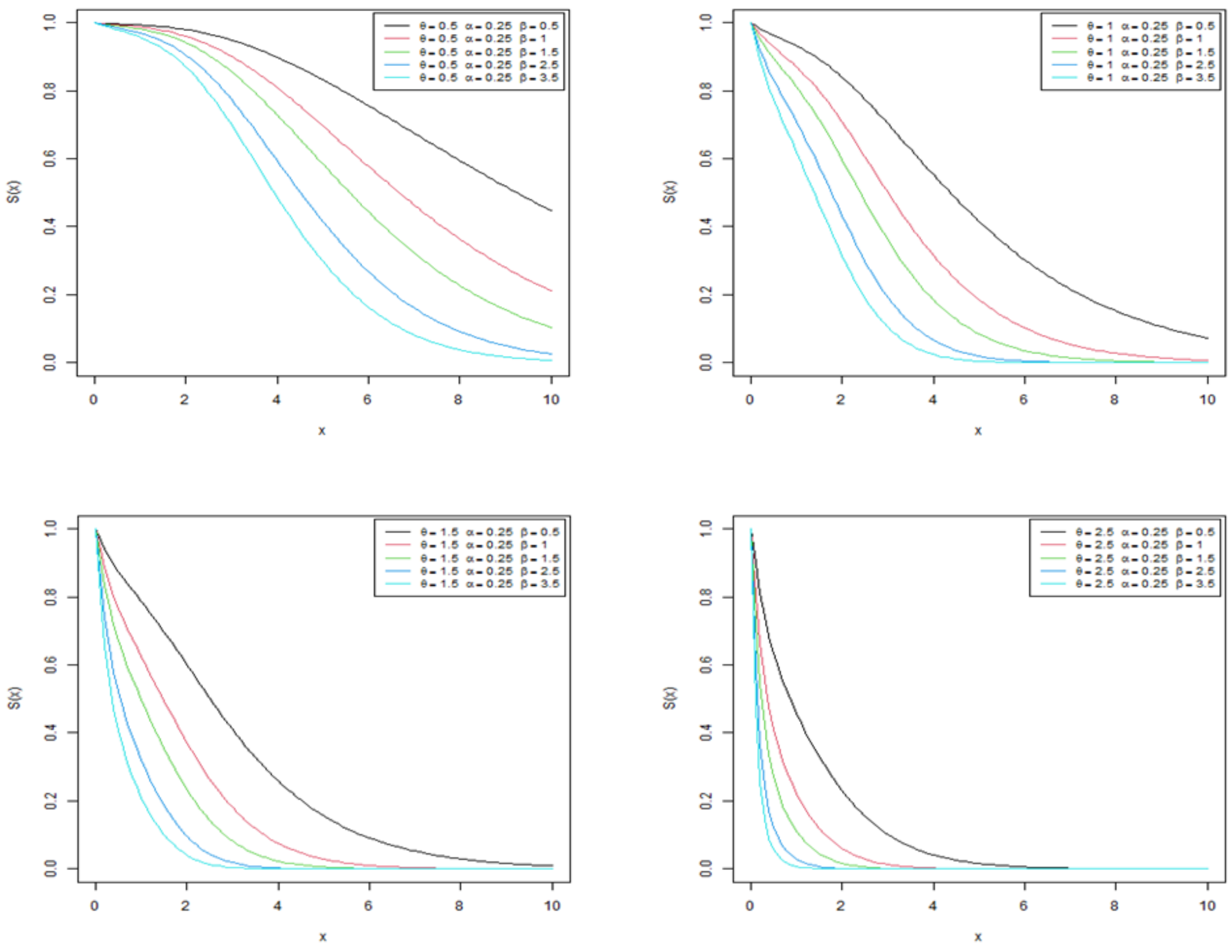
$$E(x^r) = \acute{\mu}_r = \int_0^\infty x^r\, f_{\beta\alpha}(x;\theta)\,d\,x \quad \textbf{(10)}$$

where, $r = 0, 1, 2, 3, \ldots$

$$E(x^r) = \frac{\beta\theta^4}{\theta^3+6}\int_0^\infty x^r\,(1+x^3)e^{-\theta x}\left\{1+\alpha-2\alpha\left[1-\left(\left[1+w_{(\theta)}\right]e^{-\theta x}\right)^{\beta}\right]\right\}$$
$$\left[\left[1+w_{(\theta)}\right]e^{-\theta x}\right]^{\beta-1} d\,x \quad \textbf{(11)}$$

The r^{th} non-central moment of the Hyb-R distribution can be obtained numerically.

The moment generating function of Hyb-R distribution can be obtained numerically



Figurer 4. Behavior of the SF for varying values of parameter θ, α, and β using Hyb-R distribution.

Table 1. Numerical results of the Hyb-R distribution for various parameter values of β, α, and θ.

β	α	θ	Mean	Variance	Skewness	Kurtosis	First quantile	Second quantile	Third quantile	(CV)
0.50	0.25	0.5	10.50	37.95	1.30	5.68	6.08	9.25	13.57	0.59
0.75	0.50	0.5	7.80	18.80	1.25	5.67	4.74	7.00	9.97	0.56
0.75	0.75	1.0	3.17	4.27	1.09	5.35	1.71	2.90	4.28	0.65
1.50	0.80	2.0	0.45	0.29	2.26	9.84	0.10	0.26	0.60	1.18
2.00	0.90	3.0	0.12	0.02	3.44	24.36	0.03	0.08	0.16	1.20

$$M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx}\, f_{\beta\alpha}(x;\theta)\, d\,x \tag{12}$$

where,

$$E(e^{tx}) = \frac{\beta\theta^4}{u_{2(\theta)}} \int_0^{\infty} e^{tx}\,(1+x^3)e^{-\theta x}\left\{1+\alpha-2\alpha\left[1-\left(\left[1+w_{(\theta)}\right]e^{-\theta x}\right)^{\beta}\right]\right\} \times \left[\left[1+w_{(\theta)}\right]e^{-\theta x}\right]^{\beta-1} d\,x \tag{13}$$

Table 1 presents the calculated values for the distribution properties, including the mean, variance, skewness, kurtosis, and quantiles, for a set of parameter values β, α, and θ.

4.1. Property Analysis

As β, α, and θ increase, the mean tends to decrease, indicating that the distribution shifts towards smaller values. This is logical because extreme values can significantly lower the mean β and θ increase. High variance values indicate greater variability in the data. When β is small, the variance is high, reflecting significant variability in the data. As β and θ increase, variance decreases markedly. This decrease in variance suggests that the distribution becomes more consistent and less dispersed. Higher values of β and θ lead to a more tightly clustered distribution, with less spread around the mean.

Positive skewness indicates that the distribution's tail is longer on the right side, meaning that there are more extreme high values. As β and α increase, the skewness generally increases, reflecting a shift towards higher values and a more pronounced right tail. This trend shows that with higher β and α, the distribution becomes more skewed to the right, with higher values becoming more prominent.

High kurtosis values reflect heavier tails and a more pronounced peak in the distribution. For instance, as α and θ increase, the kurtosis values rise significantly, indicating the presence of more extreme values and heavier tails. This suggests that with higher α and θ, the distribution has more pronounced tails and a greater likelihood of extreme values compared to a normal distribution.

The Q_1, Q_2, and Q_3 follow a pattern that reflects the overall distribution of values. As β, α, and θ increase, Q_1, Q_2, and Q_3 decrease. This trend indicates that as the parameters increase, the values of the quantiles shift towards smaller values. This suggests that with higher parameters, the distribution becomes more concentrated around lower values, with quantiles decreasing accordingly. Coefficient of variation (CV) measures the relative variability compared to the mean. Higher CV values indicate greater relative dispersion.

5. MAXIMUM LIKELIHOOD ESTIMATION OF HYB-R DISTRIBUTION PARAMETER UNDER SINGLE CENSORED

Cohen [46] gave the likelihood function of type-I, type-II, and type-I hybrid censored as follows Equation (14);

$$L = \frac{n!}{(n-r)!}\prod_{i=1}^{n} f\left(x_{(i)}\right)\left(1-F(y)\right)^{n-r} \tag{14}$$

where,

$$y = \begin{cases} r = n & complete\ sample \\ y = T & in\ Tpye-\text{I censored} \\ y = x_{(r)} & in\ Tpye-\text{II censored} \\ y = \min\left(T, x_{(r)}\right) & in\ Tpye-\text{I HCS} \end{cases} \tag{15}$$

Here, we shall estimate the parameters of the Hyb-R distribution. Let x_1, x_2, ... x_r constitute a random sample of size n. Then, the likelihood function is defined as

$$L \propto \prod_{i=1}^{n}\left[\frac{\beta\theta^4}{u_{2(\theta)}}(1+x^3)\,e^{-\theta x}\left\{1+\alpha-2\alpha\left[1-\left([1+w_{(\theta)}]e^{-\theta x}\right)^{\beta}\right]\right\}.\left([1+w_{(\theta)}]e^{-\theta x}\right)^{\beta-1}\right]$$
$$\times\left\{\left[1-\left([1+w_{(\theta)}]e^{-\theta y}\right)^{\beta}\right]\left\{(1+\alpha)-\alpha\left[1-\left([1+w_{(\theta)}]e^{-\theta y}\right)^{\beta}\right]\right\}\right\}^{n-r},$$

where, $u_{1(\theta)} = \theta^3y^3 + 3\theta^2y^2 + 6\theta y$, $u_{2(\theta)} = \theta^3 + 6$ and $w_{(\theta)} = \frac{u_{1(\theta)}}{u_{2(\theta)}}$.

$$L \propto \left(\frac{\beta\theta^4}{u_{2(\theta)}}\right)^{n}\prod_{i=1}^{n}\left[(1+x_{(i)}{}^3)e^{-\theta\sum_{i=1}^{n}x}\left\{1+\alpha-2\alpha\left[1-\left(\left[1+W_{(\theta)}\right]e^{-\theta x}\right)^{\beta}\right]\right\}.\left(\left[1+W_{(\theta)}\right]e^{-\theta x}\right)^{\beta-1}\right]$$
$$\times\left\{\left[1-\left(\left[1+W_{(\theta)}\right]e^{-\theta y}\right)^{\beta}\right]\left[(1+\alpha)-\alpha\left[1-\left(\left[1+W_{(\theta)}\right]e^{-\theta y}\right)^{\beta}\right]\right]\right\}^{n-r} \quad \textbf{(16)}$$

Taking the logarithm of the likelihood function in Equation (16), we obtain the log likelihood function as Equation (17).

$$\ln L = n\ln(\beta) + 4n\ln\theta - n\ln(u_{2(\theta)}) + \sum_{i=1}^{n}\ln(1+x_{(i)}{}^3) - \theta\sum_{i=1}^{n}x_{(i)} + \ln(1+\alpha)$$
$$-\ln(2\alpha) + \beta\ln(1+w_{(\theta)}) - \theta\sum_{i=1}^{n}x_{(i)} + (\beta-1)\ln(1+w_{(\theta)}) - \theta\sum_{i=1}^{n}x_{(i)}$$
$$+(n-r)\{[\beta\ln(1+w_{(\theta)}) - \theta y] + [\ln(1+\alpha) - \ln(\alpha) + \beta\ln(1+w_{(\theta)})$$
$$-\theta y]\} \quad \textbf{(17)}$$

The first derivative of the log-likelihood function with respect to β, α, and θ is given by

$$\frac{\partial\ln L}{\partial\beta} = \frac{n}{\beta} + 2[1+(n-r)]\ln(1+w_{(\theta)}),$$

$$\frac{\partial\ln L}{\partial\alpha} = \frac{1+n-r}{\alpha(1+\alpha)},$$

$$\frac{\partial\ln L}{\partial\theta} = \frac{4n}{\theta} - \frac{n\,\acute{u}_{2(\theta)}}{u_{2(\theta)}} - 3\sum_{i=1}^{n}x_{(i)} + [\beta+(\beta-1)]\frac{1}{1+w_{(\theta)}}\frac{\partial w_{(\theta)}}{\partial\theta}; + \; +2(n-r)\left[\beta.\frac{1}{1+w_{(\theta)}}.\frac{\partial w_{(\theta)}}{\partial\theta} - y\right]$$

where $\acute{u}_{1(\theta)} = 3\theta^2y^3 + 6\theta y2 + 6y$, $\acute{u}_{2(\theta)} = 3\theta^2$, $w_{(\theta)} = \frac{u_{1(\theta)}}{u_{2(\theta)}}$ and $\frac{\partial w_{(\theta)}}{\partial\theta} = \frac{\acute{u}_{1(\theta)}u_{2(\theta)} - \acute{u}_{2(\theta)}u_{1(\theta)}}{(u_{2(\theta)})^2}$

By equating $\frac{\partial\ln l}{\partial\beta}_{|\beta=\hat{\beta}}, \frac{\partial\ln l}{\partial\alpha}_{|\alpha=\hat{\alpha}}$ and $\frac{\partial\ln l}{\partial\theta}_{|\theta=\hat{\theta}}$ to zero and solving to get the MLEs $\hat{\beta}, \hat{\alpha}$ and $\hat{\theta}$, Equations (18)–(20) are obtained.

$$\frac{\partial\ln L}{\partial\hat{\beta}} = \frac{n}{\hat{\beta}} + 2[1+(n-r)]\,Ln(1+w_{(\theta)}) = 0, \quad \textbf{(18)}$$

$$\frac{\partial\ln L}{\partial\hat{\alpha}} = \frac{1+n-r}{\hat{\alpha}(1+\hat{\alpha})} = 0 \quad \textbf{(19)}$$

$$\frac{\partial\ln L}{\partial\hat{\theta}} = \frac{4n}{\hat{\theta}} - \frac{n\,\acute{u}_{2(\theta)}}{u_{2(\theta)}} - 3\sum_{i=1}^{n}x_{(i)} + [\hat{\beta}+(\hat{\beta}-1)]\frac{1}{1+w_{(\theta)}}.\frac{\partial w_{(\theta)}}{\partial\theta}$$
$$+2(n-r)\left[\hat{\beta}.\frac{1}{1+w_{(\theta)}}.\frac{\partial w_{(\theta)}}{\partial\theta} - y\right] = 0 \quad \textbf{(20)}$$

The MLEs of the parameters β, α, and θ can be obtained by solving the nonlinear equations from Equations (18)–(20) numerically.

Suppose that is $\hat{\omega}$ the MLEs of the parameter vector $\omega = (\beta, \alpha, \theta)$. The approximate variance-covariance matrix $I^{-1}(\hat{\omega})$ of $\hat{\omega}$ will be

$$I^{-1}(\hat{\omega}) = -E\left[\begin{matrix}\frac{\partial^2 LnL}{\partial\alpha^2} & \frac{\partial^2 Ln\,L}{\partial\alpha\partial\theta} & \frac{\partial^2 Ln\,L}{\partial\alpha\partial\beta}\\ \frac{\partial^2 Ln\,L}{\partial\theta\partial\alpha} & \frac{\partial^2 LnL}{\partial\theta^2} & \frac{\partial^2 Ln\,L}{\partial\theta\partial\beta}\\ \frac{\partial^2 Ln\,L}{\partial\beta\partial\alpha} & \frac{\partial^2 Ln\,L}{\partial\beta\partial\theta} & \frac{\partial^2 LnL}{\partial\theta\beta^2}\end{matrix}\right]^{-1}_{\omega=\hat{\omega}} = \begin{pmatrix}Var(\hat{\alpha}) & Cov(\hat{\alpha},\hat{\theta}) & Cov(\hat{\alpha},\hat{\beta})\\ Cov(\hat{\theta},\hat{\alpha}) & Var(\hat{\theta}) & Cov(\hat{\theta},\hat{\beta})\\ Cov(\hat{\beta},\hat{\alpha}) & Cov(\hat{\beta},\hat{\theta}) & Var(\hat{\beta})\end{pmatrix}$$

Now, the confidence interval of the vector of the unknown parameters $\omega = (\beta, \alpha, \theta)$ can be derived based on the asymptotic distribution of the MLE of

Table 2. Different schemes for type-I censoring, type-II censoring and hybrid censoring.

Estimation under	Parameters of simulation
Type-I censoring	Time of censoring or truncation (T) are determined as follows: Assumed T as a quantile from the distribution *RAM* (θ, α, β). The quantiles considered are q = 0.25, 0.50, 0.75, where: $T = F^{-1}(q, \theta, \alpha, \beta)$
Type-II censoring	Number of failure items (r) is assumed as a quantile from sample size (q_r) where q_r = 0.40, 0.60, 0.80 and $r = [q_r * n]$
Hybrid censoring	Determination of hybrid censoring parameters (x_r, T), as follows: **a. Censoring or truncation time (*T*):** Assumed as a quantile from the distribution *RAM* (θ, α, β). The quantiles considered are q = 0.25, 0.50, 0.75, where: $T = F^{-1}(q, \theta, \alpha, \beta)$ **b. Number of failure items (x_r):** Determine x_r according to the number of failure items r as a percentage of sample size (n): For n = 50, r = 15, 30, 40 For n = 100, r = 30, 60, 80. Finally, the time of censoring is the minimum of T and x_r. Based on this determination, the number of failure LE sis calculated.

Table 3. Goodness of fit test for different models.

Distribution	MLE			K-S	p-value
PHM in RTM	0.9792 (0.0930)	0.3343 (0.4685)	0.1663 (0.0521)	0.0450	0.9874
PHM	1.1429 (0.0063)	0.1581 (0.0157)	---	0.0458	0.9848
Weibull	1.4585 (0.1098)	10.9556 (0.7943)	---	0.0578	0.8919
Rama	0.4018 (0.0199)	---	---	0.1578	0.0137
Exponential	9.8786 (0.9881)	---	---	0.1730	0.0050
gamma	2.0086 (0.2638)	4.9171 (0.7331)	---	0.1945	0.0010

the parameters, as is known.

$$(\hat{\omega}-\omega) \rightarrow N_3\left(0, I^{-1}(\hat{\omega})\right)$$

where $I^{-1}(\hat{\omega})$ is the asymptotic variance-covariance matrix. Under particular regularity conditions, if the two side $100(1-\gamma)$ %, $0<\gamma<1$, the asymptotic confidence interval for the vector of the unknown parameters $\omega = (\beta, \alpha, \theta)$ can be obtained as follows Equation (21).

$$\hat{\omega} \pm c_{\frac{\gamma}{2}}\sqrt{Var(\hat{\omega})} \quad \textbf{(21)}$$

where $Var(\hat{\omega})$ is the element of the main diagonal of $I^{-1}(\hat{\omega})$ and $c_{\frac{\gamma}{2}}$ is the upper $\frac{\gamma^{th}}{2}$ percentile of the standard normal distribution.

The estimators of the asymptotic variance-covariance matrix of the parameters are obtained by inverting Fisher's information matrix, where the elements are negatives of expected values of the second partial derivatives of the logarithm of the likelihood function. The elements of the sample information matrix,

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = -\frac{(1+n-r)(1+2\alpha)}{\alpha^2(1+\alpha)^2}$$

$$\frac{\partial^2 \ln L}{\partial \theta^2} = \frac{-4n}{\theta^2} - n\left(\frac{-3\theta^4+36\theta}{\left(u_{2(\theta)}\right)^2}\right) - [\beta+(\beta-1)]\left(\frac{1}{\left(1+w_{(\theta)}\right)^2}.\left(\frac{\partial w_{(\theta)}}{\partial\theta}\right)^2\right)$$
$$+[\beta+(\beta-1)].\frac{1}{1+w_{(\theta)}}.\frac{\partial^2 w_{(\theta)}}{\partial\theta^2} - 2(n-r).\beta\left(\frac{1}{\left(1+w_{(\theta)}\right)^2}.\left(\frac{\partial w_{(\theta)}}{\partial\theta}\right)^2\right)$$
$$+2(n-r).\beta.\frac{1}{1+w_{(\theta)}}.\frac{\partial^2 w_{(\theta)}}{\partial\theta^2}$$

$$\frac{\partial^2 \ln L}{\partial \beta^2} = -\frac{n}{\beta^2}$$

$$\frac{\partial^2 \ln L}{\partial \alpha\partial\theta}, \ \frac{\partial^2 Ln\, L}{\partial \alpha\partial\beta}, \ \frac{\partial^2 Ln\, L}{\partial \theta\partial\alpha} \ and \ \frac{\partial^2 Ln\, L}{\partial \beta\partial\alpha} = 0$$

$$\frac{\partial^2 \ln L}{\partial \theta\partial\beta} = [1+(n-r)].\frac{2}{1+w_{(\theta)}}.\frac{\partial w_{(\theta)}}{\partial\theta}$$

where $\acute{u}_{1(\theta)} = \frac{\partial^2}{\partial\theta^2}u_{1(\theta)} = 6\theta y^3 + 6y^2$, $\acute{u}_{2(\theta)} = \frac{\partial^2}{\partial\theta^2}u_{2(\theta)} = 6\theta$

and

$$\frac{\partial^2}{\partial\theta^2}w_{(\theta)} = \frac{\left(u_{2(\theta)}\right)\left[\dot{u}_{1(\theta)}\dot{u}_{2(\theta)} + \acute{u}_{2(\theta)} * \frac{\partial^2 u_{1(\theta)}}{\partial\theta^2} - \dot{u}_{2(\theta)}\dot{u}_{1(\theta)} - \dot{u}_{1(\theta)}\frac{\partial^2 u_{2(\theta)}}{\partial\theta^2}\right] - 2u_{2(\theta)}\dot{u}_{2(\theta)}\left[\dot{u}_{1(\theta)}u_{2(\theta)} - \dot{u}_{2(\theta)}u_{1(\theta)}\right]}{\left(u_{2(\theta)}\right)^4}$$

6. BAYESIAN ESTIMATION

In this section, the Bayesian estimation procedure was discussed for the parameters of the Rama distribution under a random censoring scheme. We get BEs of the unknown parameters under the squared error loss (SEL) function and the general entropy loss (GEL) function. Similarly, as in Kundu and Gupta [47], Outa et al. [48], Vandana et al. [49], Gelman et al. [50], Ghosh et al. [51], Lawless [52], and Mood et al. [53], it is assumed that all the unknown parameters (θ and β) have independent gamma prior and α~uniform (a_1, b_1), can be written with proportion as follows Equations (22)–(24).

$$\prod(\alpha \setminus a_1, b_1) \propto \alpha^{a_1-1}\exp(-ab_1) \ ; \ 0 \le \alpha \le 1, a_1, b_1 > 0 \quad \textbf{(22)}$$

$$\prod(\theta \setminus a_2, b_2) \propto \theta^{a_2-1}\exp(-\theta b_2) \ ; \ \theta > 0, a_2, b_2 > 0 \quad \textbf{(23)}$$

$$\prod(\beta \setminus a_3, b_3) \propto \beta^{a_3-1}\exp(-\beta b_3) ; \ \beta > 0, a_3, b_3 > 0 \quad \textbf{(24)}$$

Therefore, the joint prior density of α, θ, and β can be written proportionally as follows Equation (25);

$$\pi\,(\alpha,\theta,\beta/x) \propto \frac{1}{\varphi}\,\pi(\alpha,\theta,\beta)\,L(\alpha,\theta,\beta/x) \quad \textbf{(25)}$$

where . Therefore, the Bayes estimator of any function of α, θ, and β say $\phi\,(\alpha, \theta, \beta)$, under the SE loss function is Equation (26).

$$\tilde{\phi}_{SE}(\alpha,\theta) = \frac{1}{\varphi}\int_0^\infty \int_0^\infty \int_0^1 \phi(\alpha,\theta,\beta)\pi(\alpha,\theta,\beta)\, L(\alpha,\theta,\beta/x)\, d\alpha\, d\theta d\beta \quad (26)$$

The Bayes estimator under the LINEX loss function of any function ϕ (α, θ, β) is the posterior mean, which is given by Equation (27);

$$\tilde{\phi}_{\text{LINEX}}(\alpha,\theta,\beta) = -\frac{1}{\upsilon}\ln\left[E_{\alpha,\theta,\beta/x}\ \left[e^{-\upsilon\phi(\alpha,\theta,\beta)}\right]\right], \quad \upsilon \neq 0$$

where

$$E_{\alpha,\beta,\beta/x}\left[e^{-\upsilon\phi(\alpha,\theta,\beta)}\right] = \frac{1}{\varphi}\int_0^\infty \int_0^\infty \int_0^1 e^{-\upsilon\,\phi(\alpha,\theta,\beta)}\pi(\alpha,\theta,\beta)\, L(\alpha,\theta,\beta/x)\, d\alpha\, d\theta\, d\beta \quad (27)$$

also, the Bayes estimator of ϕ (α, θ, β) using GE loss function is Equation (28);

$$\tilde{\phi}_{\text{GE}}(\alpha,\theta,\beta) = \left[E_{\alpha,\theta,\beta/x}[\phi(\alpha,\theta,\beta)^{-t}]\right]^{-\frac{1}{t}}, \quad t \neq 0$$

where

$$E_{\alpha,\beta,\beta/x}[\phi(\alpha,\theta,\beta)^{-t}] = \frac{1}{\varphi}\int_0^\infty \int_0^\infty \int_0^1 \phi(\alpha,\theta,\beta)^{-t}\,\pi(\alpha,\theta,\beta)\, L(\alpha,\theta,\beta/x)\, d\alpha\, d\theta\, d\beta \quad (28)$$

Equations from Equations (22)–(28) are hard to obtain, so the Markov Chain Monte Carlo (MCMC) approach can be suggested as an approximation of the Bayes estimates α, θ, β, and generating a posterior sampling using the MH algorithm, see Calabria and Pulcini [54], Ravenzwaaij et al. [55], Robert and Casella [56], and Dey and parahan [57].

6.1. Metropolis-Hastening Algorithm

To perform the MH algorithm for hybridization Rama distribution (HRD), a proposal distribution and initial values of the unknown parameters α, θ, and β have to be defined. For the proposal distribution, a bivariate normal distribution will be considered, that is

$$q\ (\{\alpha',\theta',\beta'\} \mid \{\alpha,\theta,\beta\}) \equiv N3(\{\alpha,\theta,\beta\}, S\{\alpha,\theta,\beta\})$$

Where S {α, θ, β} represents the variance-covariance matrix, negative observations may be obtained, which are unacceptable. For the initial values, the MLE for α, θ, and β is considered, that is

$$\{\alpha(0), \theta(0), \beta(0)\} = \{\hat{\alpha}, \hat{\theta}, \hat{\beta}\}$$

The selection of S {α, θ, β} is considered to be the asymptotic variance-covariance matrix I^{-1} {$\hat{\alpha}$, $\hat{\theta}$, $\hat{\beta}$}, where I {.} is the Fisher information matrix. It is noticed that the selection of S {α, θ, β} is an important issue in the MH algorithm, where the acceptance rate depends upon this. Samples from the proposal distribution are not accepted automatically as posterior samples, these candidate samples are accepted probabilistically based on the acceptance probability. more clearly for the steps of MH algorithm to draw a sample, follow the following steps: Step 1. Set the initial value of ω as $\omega^{(0)} = \{\hat{\alpha}, \hat{\theta}, \hat{\beta}\}$. Step 2. For i = 1, 2, ..., M

Table 4. Point estimation under different censoring types for dataset.

Scheme Method of Estimation		Type-I censoring $T = 10$	Type-I censoring $T = 20$	Type-II censoring $r = 40$	Type-II censoring $r = 80$	Hybrid censoring {$T = 8$, $r = 30$}	Hybrid censoring {$T = 8$, $r = 60$}	Complete
MLE	θ	0.94203	0.98224	0.78696	0.95535	0.60630	0.88062	0.97859
	α	0.41005	0.38313	0.47339	0.13392	0.39369	0.38187	0.33301
	β	0.17215	0.15958	0.25813	0.20440	0.56320	0.20663	0.16634
BE-SE	θ	1.73363	1.48934	0.80960	0.77369	0.52074	0.62849	1.56942
	α	0.27226	0.18279	0.40926	0.09211	0.14602	0.28895	0.33451
	β	0.06887	0.09570	0.27308	0.31345	1.13635	0.45945	0.10950
BE-LN-1 (υ = 0.5)	θ	1.71827	1.47312	0.80502	0.77048	0.51822	0.62671	1.55562
	α	0.27129	0.18175	0.40182	0.09153	0.14526	0.28790	0.33394
	β	0.06878	0.09554	0.27177	0.31213	1.09088	0.45503	0.10902
BE-LN-1 (υ = -0.5)	θ	1.74900	1.50553	0.81422	0.77702	0.52338	0.63026	1.58439
	α	0.27323	0.18384	0.41722	0.09269	0.14680	0.28998	0.33508
	β	0.06896	0.09586	0.27441	0.31475	1.17948	0.46406	0.10999

Table 5. Interval estimation under different censoring types for dataset.

Scheme Method of Estimation		Type-I censoring		Type-II censoring		Hybrid censoring		Complete
		$T = 10$	$T = 20$	$r = 40$	$r = 80$	$\{T = 8, r = 30\}$	$\{T = 8, r = 60\}$	
Asy-CI Lower	θ	0.44000	0.51878	0.28553	0.36770	0.05326	0.35566	0.54769
	α	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
	β	0.00000	0.00000	0.00000	0.02803	0.00000	0.00000	0.04606
Asy-CI Upper	θ	1.44406	1.44570	1.28839	1.54299	1.15934	1.40558	1.40950
	α	2.80230	1.77005	1.43781	1.50834	1.62451	4.22504	1.32629
	β	0.51738	0.32734	0.53412	0.38078	0.27006	0.83488	0.28662
HPD Lower	θ	0.76985	0.84344	0.58865	0.61736	0.39887	0.47151	0.91379
	α	0.17759	0.05836	0.16467	0.03575	0.05804	0.15237	0.24576
	β	0.04017	0.06091	0.15816	0.17776	0.26571	0.17802	0.05629
HPD Upper	θ	2.13043	1.86851	1.02961	1.00718	0.84741	0.91066	2.11919
	α	0.43110	0.43680	0.78248	0.19219	0.31696	0.39058	0.42644
	β	0.18027	0.17222	0.40436	0.41504	1.68128	0.71602	0.20124

repeat the following steps: 2.1: Set $\omega = \omega^{(i-1)}$, 2.2: Generate a new candidate parameter value δ from $N_3(\ln \omega, S_\omega)$, 2.3: Set $\omega' = \exp(\delta)$, 2.4: Using the candidate point ω', calculate $B = \frac{\pi(\acute{\omega}|x)}{\pi(\omega|x)}$ where π (.) is the posterior density in Eq. (19), 2.5: Generate a sample u from the uniform U (0,1) distribution, 2.6: Accept or reject the new candidate ω'

$$\begin{cases} \text{if } u \leq B \text{ set } \omega^{(i)} = \vartheta' \\ \text{otherwise set } \omega^{(i)} = \vartheta. \end{cases}$$

Finally, from the random samples of size M drawn from the posterior density, some of the initial samples can be discarded (burn-in), and the remaining samples can be further carried out to calculate Bayesian estimates. More accurately, from Equation (23) can be estimated as

$$\tilde{g}_{MH}(\alpha, \theta, \beta) = \frac{1}{M - L_B} \sum_{i=L_B}^{M} g(\alpha_i, \theta_i, \beta_i)$$

Where L_B represent the number of born-in samples.

6.2. Highest Posterior Density Interval

In this sub-section, HPD credible intervals for the unknown parameters α, θ, and β of the HRD based on complete samples are constructed using the samples drawn from the proposed MH algorithm in the previous sub-section. Suppose that $\alpha^{(\gamma)}$, $\theta^{(\gamma)}$, $\beta^{(\gamma)}$ are the γ^{th} quantile of α, θ, and β, respectively, that is,

$$\{\alpha(\Upsilon), \theta(\Upsilon), \beta(\Upsilon)\} = \inf\{\{\alpha, \theta\}: \prod(\{\alpha, \theta, \beta\}|x) \geq \Upsilon\},$$

where $\prod(.)$ is the posterior distribution function of α, θ, and β, $0 < \gamma < 1$. Notice that for a given α^*, θ^*, and β^*, a simulation consistent estimator of π $(\{\alpha, \theta, \beta\}|x)$ can be estimated as

$$\prod(\{\alpha^*, \theta^*, \beta^*\}|x) = \frac{1}{M - L_B} \sum_{i=L_B}^{M} I\{\alpha, \theta, \beta\} \leq \{\alpha^*, \theta^*, \beta^*\}$$

Here I $\{\alpha, \theta, \beta\} \leq \{\alpha^*, \theta^*, \beta^*\}$ is the indicator function. Then the corresponding estimate is obtained as

$$\widehat{\prod}(\{\alpha^*, \theta^*, \beta^*\}|x) = \begin{cases} 0 & \text{if } \{\alpha^*, \theta^*, \beta^*\} < \{\alpha_{(L_B)}, \theta_{(L_B)}, \beta_{(L_B)}\} \\ \sum_{H=L_B}^{i} \omega_H & \text{if } \{\alpha_{(i)}, \theta_{(i)}, \beta_{(i)}\} < \{\alpha^*, \theta^*, \beta^*\} < \{\alpha_{(i+1)}, \theta_{(i+1)}, \beta_{(i+1)}\} \\ 1 & \text{if } \{\alpha^*, \theta^*, \beta^*\} > \{\alpha_{(M)}, \theta_{(M)}, \beta_{(M)}\} \end{cases}$$

Where, $\omega_H = \frac{1}{M - L_B}$ and $\{\alpha_{(H)}, \theta_{(H)}, \beta_{(H)}\}$ are the order values of $\{\alpha_{(H)}, \theta_{(H)}, \beta_{(H)}\}$ Now, for H = L_B, I, M, $\{\alpha^{(\gamma)}, \theta^{(\gamma)}, \beta^{(\gamma)}\}$ can be approximated by

$$\{\tilde{\alpha}(\Upsilon), \tilde{\theta}(\Upsilon), \tilde{\beta}(\Upsilon)\} = \begin{cases} \{\alpha_{(L_B)}, \theta_{(L_B)}, \beta_{(L_B)}\} & \text{if } \Upsilon = 0 \\ \{\alpha_{(i)}, \theta_{(i)}, \beta_{(i)}\} & \text{if } \sum_{H=L_B}^{i-1} \omega_H < \Upsilon < \sum_{H=L_B}^{i} \omega_H \end{cases}$$

Now to obtain a 100(1 – Υ) % HPD credible interval for α, θ, β let

$$HPD^{\alpha}{}_{H} = \left(\tilde{\alpha}^{\left(\frac{H}{M}\right)}, \tilde{\alpha}^{\left(\frac{H+(1-\Upsilon)M}{M}\right)}\right), HPD^{\theta}{}_{H} = \left(\tilde{\theta}^{\left(\frac{H}{M}\right)}, \tilde{\theta}^{\left(\frac{H+(1-\Upsilon)M}{M}\right)}\right)$$

and

$$HPD^{\beta}{}_{H} = \left(\tilde{\beta}^{\left(\frac{H}{M}\right)}, \tilde{\beta}^{\left(\frac{H+(1-\Upsilon)M}{M}\right)}\right),$$

for $H = L_B, \ldots, M$. Then choose HPD_{H^*} among all the HPD_H's such that it has the smallest width.

7. SIMULATION STUDY AND REAL DATA ANALYSIS

In this section, we aim to present and analyze the results of a simulation study conducted to evaluate the performance of MLEs and Bes. This is achieved through two illustrative examples: the first based on Monte Carlo simulated data, and the second involving a real-life dataset representing the waiting times (in minutes) for 100 bank customers. The practical implementation of the theoretical findings developed in earlier chapters is carried out using the R statistical programming language. The section concludes with a summary of the main findings and recommendations.

7.1. Simulation Study

In this section, we aim to analyze the performance of the various estimation methods presented in Sections (2) and (3). To achieve this, we conduct simulation studies to evaluate the finite-sample behavior of the proposed methods under different sample sizes and censoring schemes. Both MLEs and Bes, based on the SE and LINEX loss functions, are compared using simulated data. Additionally, a real dataset is used to further illustrate and validate the inferential results established in the preceding sections. All computations have been carried out using the R statistical programming language. In particular, the MLEs and the Bayesian HPD intervals were computed using the HD Interval package in R. A simulation study was carried to check the performance of the accuracy of point and interval estimates for several cases, for which estimate the three parameters of distribution (α, θ, and β) for number of replications (m = 1000), for different sample sizes (n) as n = 25, 50, 75, 100 and different parameters values. All the computations are performed using the statistical software R.

The simulations results for MLEs are summarized in Appidex (Tables S1–S8**)** and obtained by the following steps: (i) specify initial values for parameters (α, θ, and β) as (0.75, 0.50, 1.50), (0.25, 1, 1) and (o.50, 2.50, 0.50); (ii) specify the sample size n. As n = 25, 50, 75, 100; (iii) obtain the ML estimates; (iv) obtain the average (Avg), MSE, asymptotic, and Cis for the unknown parameters, AILs, and CP for the different sample size.

The simulation results for Bayesian estimates are summarized in Supplementary material (Tables S1 –S8), which obtained by the following steps: (i) step i, and ii of the MLEs simulation are the same; (ii) by using M-H algorithm shown in Section (4.1). under the informative prior and the non-informative prior and repeat the chain N times (N = 10000) to obtain MCMC samples. For informative prior case (IF), we compute the hyper parameters for all simulation $a_1 = a_2 = a_3 = 1.5$, $b_1 = b_2 = b_3 = 2.5$, for non-informative prior case (NIF) we

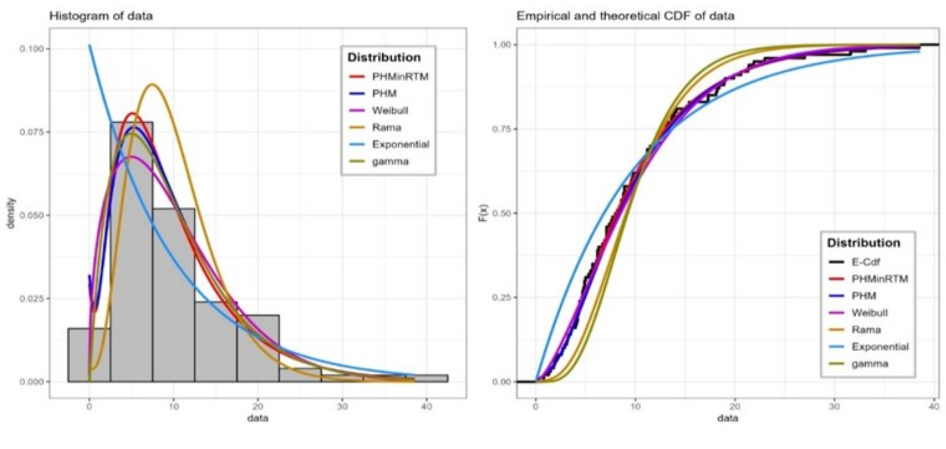


Figure 5. Plots of the estimated PDFs and CDFs for selected models for the data.

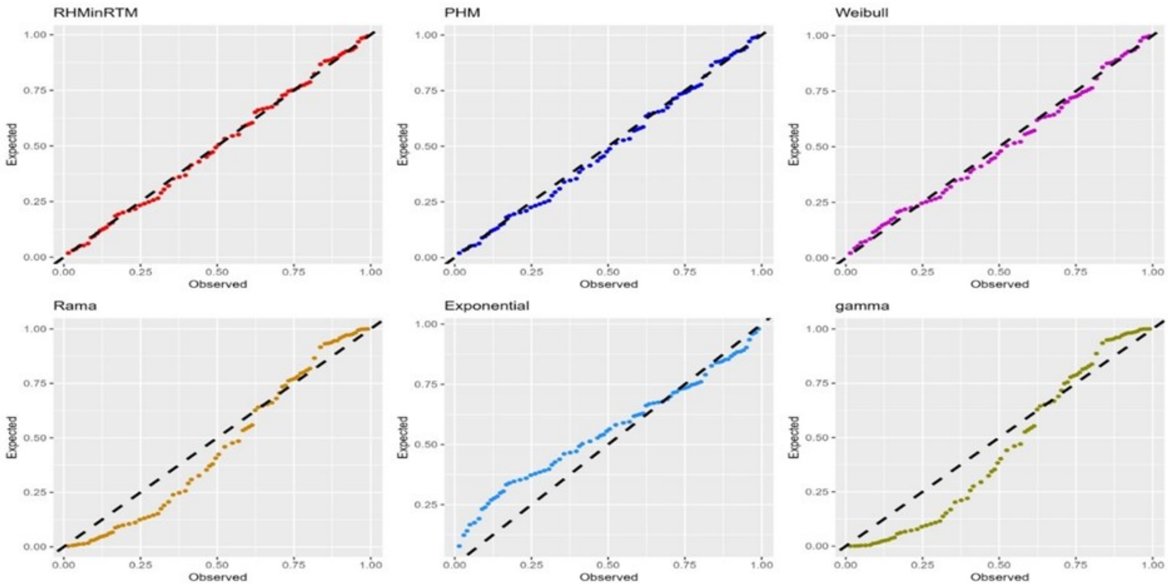


Figure 6. The PP plots of the different models for the data.

assume that hyper-parameter values are $a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = 0$; (iii) compute the approximate Bayes estimator of g (α, θ, and β) under SE loss function; (iv) repeat steps i-iii 1000 times to obtain the Avg, MSE, HPD intervals for the unknown parameters, AILs, and CP for the different sample sizes; and (v) based on the generated data, MLEs and associated 95% asymptotic confidence interval (Asy-CI) and HPD interval are computed. Note that the initial guess values are considered to LE si same as the true parameter values while obtaining MLEs, for more details see Mishra et al. [58], and Deepmala et al. [59] (Table 2).

All the average estimates for both methods are reported in Supplementary material (Tables S1–S8). Further, the first row represents the average estimates and interval estimates, and in the second row, associated MSEs and AILs with CPs are reported. From tabulated values it can be noticed that depending on MSEs, higher values of n lead to better estimates. LE s also noticed that the performance of the Bayes estimates obtained under informative prior is better than MLEs. It can also be noticed that the AIL and associated CPs of HPD intervals are better than those of MLEs. Also, the Bayes estimates under GE loss function are better than the Bayes estimates under LINEX loss function. From the results in Supplementary material (Tables S1–S8), the following conclusion can be made: (i) when the sample size increases, the MSEs decrease in the MLEs and the Bayes estimates; (ii) when T increases, the MSEs decrease in the MLEs and the Bayes estimates; (iii) the MSEs of the Bes under the LINEX loss function are less than those under the SE loss function; (iv) the average interval length (AIL) in the LE sis less than the Bes under SEL and LINX loss function in all cases; and (v) the coverage probabilities (CPs) for the unknown parameters in the LE sis greater than the CPs in the Bes and are close to 100%.

7.2. Real Data

In this section, a real dataset is analyzed to illustrate the flexibility and applicability of the proposed Hyb-R distribution in modeling lifetime data. The dataset consists of the waiting times (in minutes) of 100 bank customers, originally reported by Ghitany et al. [60]. This data has been widely used in the literature to evaluate model performance in reliability and survival studies.

7.2.1. Dataset

Waiting times (min) of 100 bank customers (Ghitany et al. [60]) as: 0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7,2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2 ,4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27.0, 31.6, 33.1, 38.5.

The following distributions in Table 3 are considered for comparison: Weibull, Exponential, Gamma, PHM, RTM, and Rama. The analysis is based on the maximum likelihood method, and the goodness-of-fit is assessed using the Kolmogorov-Smirnov (K-S) statistic, log-likelihood value (L), Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC). For the Hyb-R model, the K-S statistic value is 0.0450 with a p-value of 0.9874, indicating an excellent fit. Comparisons with alternative models such as PHM (K - S = 0.0458, p = 0.9848), Weibull (K - S = 0.0578, p = 0.8919), and Rama (K - S = 0.1578, p = 0.0137) further support the superior performance of the Hyb -R distribution.

Next, we consider parameter estimation under various censoring schemes: Type-I censoring (T = 10, 20), Type-II censoring (r = 40, 80), and hybrid censoring (e.g., T = 8, r = 30 or 60). The MLEs of the parameters θ, α, and β are obtained under each scheme. Bayesian estimates (BE) are also computed using the Metropolis-Hastings (MH) algorithm with non-informative priors under different loss functions: Squared Error Loss (SEL) and LINEX loss (with $\upsilon = \pm 0.5$).

For Bayesian analysis, 10,000 posterior samples are generated, with the first 2,000 considered as burn-in. The initial values for the MH algorithm are chosen as the MLEs of θ, α, and β. Posterior summaries, including Bayes estimates and 95% Highest Posterior Density (HPD) intervals, are computed according to the approach of Chen and Shao [61]. All estimation results, including point estimates, standard errors, asymptotic confidence intervals (Asy-CI), and HPD intervals, are presented in Tables 4 and 5. The findings confirm that the Hyb-R model provides stable and accurate parameter estimates, and its superiority in model fitting is validated across various censoring scenarios.

7.3. Graphical Analysis of Real Data

In this section, the fitted Hybridization Rama (Hyb-R) distribution is compared with several well-known distributions — including the PHM, Weibull, Rama, Exponential, and Gamma distributions through graphical methods.

The left panel of Figure 5 shows the histogram of the real dataset overlaid with the fitted PDFs of the considered models. The proposed Hyb-R distribution (denoted as PHM in RTM) provides the best fit to the empirical data compared to the alternative models. The Hyb-R curve closely follows the shape of the histogram, particularly in capturing the skewness and tail behavior. In contrast, the exponential and gamma models deviate notably in the lower and upper tails. Meanwhile, the right panel of Figure 5 displays the empirical cumulative distribution function (E-CDF) plotted alongside the theoretical CDFs of the fitted models. The CDF of the Hyb-R distribution almost perfectly matches the empirical CDF across the entire range of data. Other models, especially exponential and gamma, show significant discrepancies in the middle and tail regions.

Figure 6 presents the probability-probability (P–P) plots for each fitted model separately. The P–P plot for the Hyb-R distribution shows points closely aligned with the 45-degree reference line, indicating an excellent fit. The PHM and Weibull distributions also show reasonable fits but are slightly less aligned compared to the Hyb-R model. The Rama, exponential, and gamma models demonstrate more pronounced departures from the reference line, confirming their inferior performance relative to the Hyb-R model.

Figure 7 shows the total time on test (TTT) plot for the analyzed dataset. The TTT plot is a useful graphical tool to preliminarily assess the shape of the underlying hazard rate function. If the TTT curve is approximately linear (i.e., follows the 45-degree line), it suggests a constant hazard rate, indicating exponential-type behavior. If the TTT curve lies above the 45-degree line, it implies a decreasing hazard rate. Conversely, if the TTT curve lies below the 45-degree line, it indicates an increasing hazard rate. From the Figure 7, it is observed that the TTT curve is slightly convex above the 45-degree line at the beginning and concave below it toward the end. This pattern suggests that the data may exhibit a bathtub-shaped hazard rate — initially decreasing, then approximately constant, and finally increasing — which justifies the need for a highly flexible distribution like the proposed Hyb-R model to capture this complex behavior.

Overall, the graphical analysis strongly supports that the proposed Hyb-R distribution provides the best fit to the real dataset among all compared models. The flexibility introduced through the hybridization process significantly improves the model's ability to capture the complex structure of the data. To demonstrate the effectiveness and flexibility of the proposed Hyb-R distribution, a comparison is conducted with several well-known lifetime distributions commonly used in reliability and survival analysis, such as the Rama, Weibull, Gamma, and Exponential distributions.

The comparison is based on goodness-of-fit measures including the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and log-likelihood values. The results indicate that the Hyb-R distribution provides a better fit to the analyzed real and censored lifetime datasets compared to the competing models. This superior performance can be attributed to the additional flexibility introduced through the hybridization mechanism, which allows the model to capture various shapes of hazard rate functions and accommodate different levels of skewness and kurtosis. Moreover, the Bayesian estimation results further confirm the robustness of the proposed model, as the Hyb-R distribution yields lower estimation errors and narrower credible intervals under informative priors compared to classical maximum likelihood estimation. These findings highlight the practical relevance and novelty of the proposed distribution for modeling complex lifetime data.

Despite the flexibility and effectiveness of the proposed Hybridization Rama distribution, some limitations should be acknowledged. First, the model complexity increases due to the additional parameters introduced by the hybridization process, which may lead to higher computational cost in parameter estimation. Second, the performance of the Bayesian estimators depends on the choice of prior distributions, which may influence the results if inappropriate priors are selected. Finally, the current study focuses on univariate lifetime data; therefore, further extensions are required to address multivariate or high-dimensional data structures.

The paper's key findings are as follows: first, that larger samples produce better results from the MLEs and BEs; second, that longer censoring times yield better results from both estimators. So, in terms of reduced MLEs, the BEs that operate under the LINEX loss function outperform the SE loss function. Then, in every instance, the AIL derived from MLEs is shorter than the AIL derived from BEs when considering the SE loss function and the LINEX loss function simultaneously. Last but not least, the CPs of the MLEs are higher than those of the BEs.

It is possible to expand the scope of the future work to include: Initially, alternative loss functions can be used when using Bayesian estimation, other than the squared error loss function. After that, it is possible to look into the best practices for designing progressive Type-I censoring sampling plans. Therefore, the Hby-Rama distribution is compatible with different censoring systems. This means that other prior distributions can be used for Bayesian estimation as well. Once we have the filtering methodology proposed, we may estimate the information measures of the Hby-Rama distribution using well-established methodologies. Refer to Abu

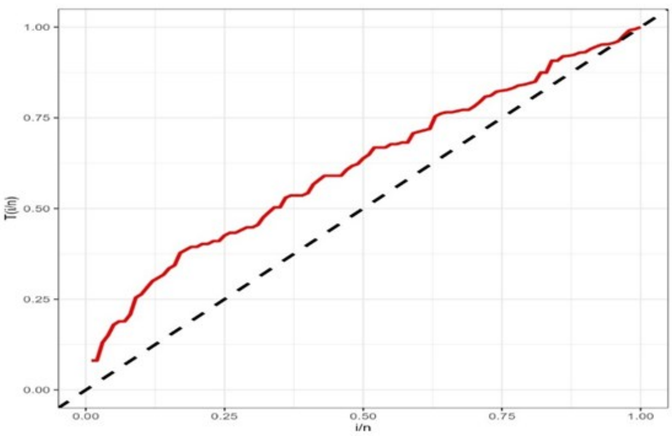


Figure 7. Estimated TTT plots for the data.

El Azm et al. [62], Alrashidi et al. [63], Nassr et al. [64]-[66], Ahmed et al. [67], Al Mutairi et al. [68], Yousef et al. [69] and Nassr and Elharoun [70] for further details on these approaches and estimation.

8. CONCLUSIONS

In this paper, the suggested hybridisation Rama (Hyb-R) distribution shows a lot of versatility when it comes to modelling lifetime data. It can handle different shapes of hazard rates and tail behaviours. Theoretical properties were obtained in closed or numerical formats, and both maximum likelihood and Bayesian estimation techniques were examined. The present study concentrates on complete data scenarios, and the model's efficacy in substantially censored or abbreviated datasets remains underinvestigated. We derived the maximum likelihood estimators (MLEs) and Bayesian estimators under various loss functions for complete and censored data, including type-I, type-II, and hybrid censoring schemes. The asymptotic confidence intervals and posterior HPD intervals were constructed for the unknown parameters. Furthermore, the applicability of the proposed model was demonstrated using a real dataset, showing that the Hyb-R model provides a better fit and more accurate estimates compared to several classical models. All proposed estimates were found to be close to the true parameter values, with associated standard errors being small. The Bayesian estimators, in particular, showed robustness and stability under different censoring scenarios, confirming the usefulness and flexibility of the Hyb-R distribution in modeling lifetime and reliability data. Lifetime models, including exponential, Weibull, gamma, and Rama distribution extensions, were compared to the Hyb-R distribution. Common goodness-of-fit metrics were used to compare models. The suggested model consistently fits all datasets better than competitors, demonstrating its flexibility and practicality.

AUTHOR INFORMATION

Corresponding Author

Eslam Abdelhakim Seyam — Department of Insurance and Risk Management, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh-13318 (Saudi Arabia);
orcid.org/0000-0002-2487-5106
Email: isiam@imamu.edu.sa

Authors

Waleed M. Afify — Department of Statistics, Mathematics and Insurance, Kafrelsheikh University, Kafrelsheikh-33511 (Egypt);
orcid.org/0000-0002-4429-4210
Ahmed E. Khairalla — Department of Statistics, Mathematics and Insurance, Kafrelsheikh University, Kafrelsheikh-33511 (Egypt);
orcid.org/0009-0005-8183-8028
Emadeldin I. A. Ali — Department of Mathematics, Statistics, and Insurance, Ain Shams University, Cairo-11566 (Egypt);
orcid.org/0009-0008-6914-181X
Said G. Nassr — Department of Statistics and Insurance, Arish University, El-Arish-45511 (Egypt);
orcid.org/0000-0002-0126-2868
Ahmed Ramzy Shehata — Department of Statistics, Mathematics and Insurance, Kafrelsheikh University, Kafrelsheikh-33511 (Egypt);
orcid.org/0009-0000-1586-9479

Author Contributions

W. M. A.: Writing—original draft, Validation, Methodology, Formal analysis, Conceptualization, Writing—review, Software. E. A. S.: Writing—original draft, Methodology, Writing—review, Investigation. A. E. K.: Writing—original draft, Validation, Methodology, Formal analysis, Software, Conceptualization, Writing—review, Investigation, Data curation. E. I. A. A.: Writing—original draft, Methodology, Writing—review, Formal analysis, Software. S. G. N.: Writing—original draft, Methodology, Writing—review, Formal analysis, Software. A. R. S.: Methodology, Formal analysis, Conceptualization, Writing—review, Investigation. Finally, all authors reviewed the results and approved the final version of the manuscript.

Conflicts of Interest

The authors declare no conflicts of interest to report regarding the present study.

SUPPORTING INFORMATION

Supplementary data associated with this article can be found in the online version at doi: 10.47352/jmans.2774-3047.430

DECLARATION OF GENERATIVE AI

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

REFERENCES

[1] R. D. Gupta and D. Kundu. (2001). "Exponentiated Exponential Family: An Alternative to Gamma and Weibull Distributions". *Biometrical Journal.* **43** (1): 117-130. 10.1002/1521-4036(200102)43:1<117::AID-BIMJ117>3.0.CO;2-R.

[2] R. D. Gupta and D. Kundu. (2009). "Introduction of Shape/Skewness Parameter (s) in a Probability Distribution". *Journal of Probability and Statistical Science.* **7** (2): 153-171.

[3] R. D. Gupta and D. Kundu. (2007). "Generalized Exponential Distribution: Different Method of Estimation". *Journal of Statistical Computation and Simulation.* **69** (4): 315-337. 10.1080/00949650108812098.

[4] D. R. Cox. (1972). "Regression Models and Life Tables". *Journal of the Royal Statistical Society: Series B (Methodological).* **34** : 187-200. 10.1111/j.2517-6161.1972.tb00899.x.

[5] S. Nadarajah and S. Kotz. (2003). "The Exponentiated Fréchet Distribution". *InterStat Electronic Journal.* 1-7.

[6] S. Nadarajah and S. Kotz. (2003). "On Some Beta-Generated Distributions". *Statistics and Probability Letters.* **65** (4): 379-384. 10.1016/j.spl.2003.07.013.

[7] S. Nadarajah and S. Kotz. (2004). "The Beta Gumbel Distribution". *Mathematical Problems in Engineering.* **2004** (4): 323-332. 10.1155/S1024123X04403068.

[8] G. Srinivasa Rao, K. Srinivasa Rao, and D. Ramesh. (2015). "A New Exponentiated Log-Logistic Distribution: Properties and Applications". *International Journal of Statistics and Applications.* **5** (6): 284-291.

[9] G. S. Mudholkar and D. K. Srivastava. (1993). "Exponentiated Weibull Family for Analyzing Bathtub Failure-Rate Data". *IEEE Transactions on Reliability.* **42** (2): 299-302. 10.1109/24.229504.

[10] D. T. Shirke and C. S. Kakade. (2006). "On Exponentiated Lognormal Distribution". *International Journal of Agricultural and Statistical Sciences.* **2** : 319-326.

[11] D. T. Shirke and R. V. Kakade. (2013). "Exponentiated Lognormal Distribution and Its Application". *Journal of Reliability and Statistical Studies.* **6** (1): 91-98.

[12] M. Sarhan and D. Kundu. (2008). "Generalized Linear Failure Rate Distribution". *Communications in Statistics: Theory and Methods.* **38** : 642-660. 10.1080/03610920802272414.

[13] R. D. Gupta and D. Kundu. (2010). "Generalized Logistic Distributions". *Journal of Applied Statistical Science.* **18** : 51-66.

[14] A. S. Hassan, M. Elgarhy, S. G. Nassr, Z. Ahmed, and S. Alrajhi. (2019). "Truncated Weibull Fréchet Distribution: Statistical Inference and Applications". *Journal of Computational and Theoretical Nanoscience.* **16** (1): 1-9. 10.1166/jctn.2019.7734.

[15] A. S. Hassan, S. E. Hemeda, and S. G. Nassr. (2020). "On the Extended Exponentiated Pareto Distribution". *Journal of Modern Applied Statistical Methods.* **19** (1). 10.22237/jmasm/1619481840.

[16] A. S. Hassan and S. G. Nassr. (2020). "A New Generalized Power Function Distribution: Properties and Estimation Based on Censored Samples". *Thailand Statistician.* **18** (2): 215-234.

[17] A. S. Hassan, M. A. Khaleel, and S. G. Nassr. (2021). "Transmuted Topp-Leone Power Function Distribution: Theory and Applications". *Journal of Statistics Applications and Probability.* **10** (1): 215-227. 10.18576/jsap.

[18] T. A. Abushal, A. S. Hassan, A. R. El-Saeed, and S. G. Nassr. (2021). "Power Inverted Topp-Leone Distribution in Acceptance Sampling Plans". *CMC: Computers, Materials and Continua.* **67** (1): 991-1011. 10.32604/cmc.2021.014620.

[19] A. S. Hassan and S. G. Nassr. (2021). "Parameter Estimation of an Extended Inverse Power Lomax Distribution with Type I Right Censored Data". *Communications for Statistical Applications and Methods.* **28** (2): 99-118. 10.29220/CSAM.2021.28.2.099.

[20] A. A. H. Ahmadini, A. S. Hassan, M. Elgarhy, M. Elsehetry, S. S. Alshqaq, and S. G. Nassr. (2021). "Inference of Truncated Lomax Inverse Lomax Distribution with Applications". *Intelligent Automation and Soft Computing.* **29** (1): 199-212. 10.32604/iasc.2021.017890.

[21] N. Sharma, L. N. Mishra, S. N. Mishra, and V. N. Mishra. (2021). "Empirical Study of New Iterative Algorithm for Generalized Nonexpansive Operators". *Journal of Mathematics and Computer Science.* **25** : 284 -295. 10.22436/jmcs.025.03.07.

[22] S. G. Nassr, A. S. Hassan, R. Alsultan, and A. R. El-Saeed. (2022). "Acceptance Sampling Plans for the Three-Parameter Inverted Topp-Leone Model". *Mathematical Biosciences and Engineering.* **19** (12): 13628 -13646. 10.3934/mbe.2022636.

[23] A. R. Gairola, S. Maindola, L. Rathour, L. N. Mishra, and V. N. Mishra. (2022). "Better Uniform Approximation by New Bivariate Bernstein Operators". *International Journal of Analysis and Applications.* **20** : 60. 10.28924/2291-8639-20-2022-60.

[24] A. Younus, Z. Dastgeer, L. Rathour, L. N. Mishra, V. N. Mishra, and S. Pandey. (2022). "Distinguishability Criteria of Conformable Hybrid Linear Systems". *Nonlinear Engineering.* **11** (1): 420-427. 10.1515/nleng-2022-0045.

[25] M. Kamal, M. M. Alsolmi, Nayabuddin, A. Al Mutairi, E. Hussam, M. S. A. Mustafa, and S. G. Nassr. (2023). "A New Distributional Approach: Estimation, Monte Carlo Simulation and Applications to the Bio -Medical Data Sets". *Networks and Heterogeneous Media.* **18** (4): 1575-1599. 10.3934/nhm.2023069.

[26] M. Kamal, R. Aldallal, S. G. Nassr, A. Al Mutairi, M. Yusuf, M. S. A. Mustafa, M. M. Alsolmi, and E. M. Almetwally. (2023). "A New Improved Form of the Lomax Model: Its Bivariate Extension and an Application in the Financial Sector". *Alexandria Engineering Journal.* **75** : 127-138. 10.1016/j.aej.2023.05.027.

[27] A. R. El-Saeed, A. S. Hassan, N. M. Elharoun, A. Al Mutairi, R. H. Khashab, and S. G. Nassr. (2023). "A Class of Power Inverted Topp-Leone Distribution: Properties, Different Estimation Methods and Applications". *Journal of Radiation Research and Applied Sciences.* **16** (4): 1-18. 10.1016/j.jrras.2023.100643.

[28] A. Al Mutairi, A. S. Hassan, S. S. Alshqaq, R. Alsultan, A. M. Gemeay, S. G. Nassr, and M. Elgarhy. (2023). "Inverse Power Ramos-Louzada Distribution with Various Classical Estimation Methods and Modeling to Engineering Data". *AIP Advances.* **13** (9): 095117-1-095117-22. 10.1063/5.0170393.

[29] X. Tang, J. T. Seong, R. Alharbi, A. Al Mutairi, and S. G. Nassr. (2024). "A New Probabilistic Model: Theory, Simulation and Applications to Sports and Failure Times Data". *Heliyon.* **10** (4): e25651. 10.1016/j.heliyon.2024.e25651.

[30] A. A. M. Mahmoud, R. H. Khashab, Z. I. Kalantan, S. M. S. Binhimd, A. S. Alghamdi, and S. G. Nassr. (2024). "On Bivariate Compound Exponentiated Survival Function of the Beta Distribution: Estimation and Prediction". *Journal of Radiation Research and Applied Sciences.* **17** (2): 1-18. 10.1016/j.jrras.2024.100886.

[31] M. N. Atchadé, M. A. G. N'Bouké, A. M. Djibril, A. Al Mutairi, M. S. A. Mustafa, E. Hussam, H. Alsuhabi, and S. G. Nassr. (2024). "A New Topp-Leone Kumaraswamy Marshall-Olkin Generated Family of Distributions with Applications". *Heliyon.* **10** (2): e24001. 10.1016/j.heliyon.2024.e24001.

[32] A. A. Ahmed, E. A. Seyam, A. R. El-Saeed, E. O. Abdalla, S. Naserelden, and S. G. Nassr. (2026). "A Generalized Lifetime Model Based on Ailamujia Distribution: Statistical Properties, Different Inference Estimation and Applications to Several Fields". *Scientific African.* **31** : e03179. 10.1016/j.sciaf.2026.e03179.

[33] A. A. M. Mahmoud, E. A. Seyam, R. H.

Khashab, E. Alshawarbeh, Z. I. Kalantan, O. F. Khalil, A. R. El-Saeed, and S. G. Nassr. (2025). "A New Extension of Generalized Beta Distribution: Statistical Properties, Prediction Estimation and Application in Failure Components and Physics Data". *Scientific African.* **30** : e03037. 10.1016/j.sciaf.2025.e03037.

[34] R. Shanker. (2017). "The Rama Distribution and Its Applications". *International Journal of Statistics and Applications.* **7** (3): 119-124.

[35] A. Vijayakumar, T. Muthulakshmi, and S. Ramalingam. (2020). "A Modified Kumaraswamy Distribution: Theory and Applications". *Journal of Applied Probability and Statistics.* **15** (2): 1-18.

[36] A. Vijayakumar, T. Muthulakshmi, and S. Ramalingam. (2020). "A Modified Rama Distribution: Properties and Applications". *Journal of Applied Probability and Statistics.* **15** (2): 25-40.

[37] A. A. Abeb and T. B. Getachew. (2019). "On the Generalized Rama Distribution with Applications". *Open Journal of Statistics.* **9** (3): 204-215.

[38] S. O. Chrisogonus, N. O. Eze, and O. C. Ugwoke. (2020). "Extension of the Rama Distribution Using Transmutation Technique". *Asian Journal of Probability and Statistics.* **7** (3): 35-47.

[39] A. Alzaatreh, C. Lee, and F. Famoye. (2013). "A New Method for Generating Families of Continuous Distributions". *METRON.* **71** (1): 63-79. 10.1007/s40300-013-0007-y.

[40] G. R. Aryal and C. P. Tsokos. (2009). "On the Transmuted Distributions with Applications". *Nonlinear Analysis: Theory, Methods and Applications.* **71** (12): e1401-e1407. 10.1016/j.na.2009.01.168.

[41] B. T. Ayele and A. Alemu. (2021). "A New Class of Transmuted Distributions with Applications". *SpringerPlus.* **10** (1): 122.

[42] D. H. Abdel Hady. (2019). "Use of Exponential Distribution for Hybridization of Distributions". *Advances and Applications in Statistics.* **58** (1): 57-75. 10.17654/AS058010057.

[43] W. M. Afify. (2017). "Hybridization of Distributions Using Transmutation and Proportional Hazard Approach". *Advances and Applications in Statistics.* **51** (6): 445-468. 10.17654/AS051060445.

[44] C. K. Onyekwere, A. O. George, and S. U. Enogwe. (2021). "Exponentiated Rama Distribution: Properties and Application". *Mathematical Theory and Modelling.* **11** (1).

[45] G. Srinivasa Rao, L. R. R. Kantam, K. Rosaiah, and V. S. V. Prasad. (2012). "Reliability Test Plans for Type-II Exponentiated Log-Logistic Distribution". *Journal of Reliability and Statistical Studies.* **5** (1): 55-64.

[46] A. C. Cohen. (1965). "Maximum Likelihood Estimation in the Weibull Distribution Based on Complete and on Censored Samples". *Technometrics.* **7** (4): 579-588. 10.2307/1266397.

[47] D. Kundu and R. D. Gupta. (2017). "Bayesian Estimation Under Different Loss Functions". *Communications in Statistics: Theory and Methods.* **46** (5): 2207-2227.

[48] R. Outa, F. R. Chavarette, A. C. Gonçalves, S. L. da Silva, V. N. Mishra, A. R. Panosso, and L. N. Mishra. (2021). "Reliability Analysis Using Experimental Statistical Methods and AIS: Application in Continuous Flow Tubes of Gaseous Medium". *Acta Scientiarum. Technology.* **43** e55825. 10.4025/actascitechnol.v43i1.55825.

[49] Vandana, Deepmala, K. Drachal, and L. N. Mishra. (2021). "Forecasting Art Prices with Bayesian Models". *Thai Journal of Mathematics.* **19** (2): 479-491.

[50] A. Gelman, J. B. Carlin, H. S. Stern, D. B. Dunson, A. Vehtari, and D. B. Rubin. (2013). "Bayesian Data Analysis". CRC Press. 10.1201/b16018.

[51] M. Ghosh, M. Delampady, and T. Samanta. (2006). "An Introduction to Bayesian Analysis: Theory and Methods". Springer.

[52] J. F. Lawless. (2011). "Statistical Models and Methods for Lifetime Data". Wiley.

[53] A. M. Mood, F. A. Graybill, and D. C. Boes. (1974). "Introduction to the Theory of Statistics". McGraw-Hill.

[54] R. Calabria and G. Pulcini. (1994). "An Engineering Approach to Bayes Estimation for the Weibull Distribution".

Microelectronics Reliability. **34** (5): 789-802. 10.1016/0026-2714(94)90004-3.

[55] D. van Ravenzwaaij, P. Cassey, and S. D. Brown. (2018). "A Simple Introduction to Markov Chain Monte Carlo Sampling". *Psychonomic Bulletin and Review.* **25** : 143-154. 10.3758/s13423-016-1015-8.

[56] C. P. Robert and G. Casella. (2010)." Introducing Monte Carlo Methods with R". Springer. 10.1007/978-1-4419-1576-4.

[57] S. Dey and B. Pradhan. (2014). "Generalized Inverted Exponential Distribution Under Hybrid Censoring". *Statistical Methodology.* **18** : 101-114. 10.1016/j.stamet.2013.07.007.

[58] V. N. Mishra, L. N. Mishra, N. Subramanian, and S. A. A. Abdulla. (2020). "Analytic Weighted Rough Statistical Convergence with Rate of Rough Convergence and Voronovskaya Theorem of Triple Difference Sequences". *Applied Sciences.* **22** : 157-168.

[59] Deepmala, N. Subramanian, and L. N. Mishra. (2017). "The Cesáro Lacunary Ideal Bounded Linear Operator of χ^2-of φ-Statistical Vector Valued Defined by a Bounded Linear Operator of Interval Numbers". *Songklanakarin Journal of Science and Technology.* **39** (4): 549-563. 10.14456/sjst-psu.2017.60.

[60] M. E. Ghitany, B. Atieh, and S. Nadarajah. (2008). "Lindley Distribution and Its Application". *Mathematics and Computers in Simulation.* **78** (4): 493-506. 10.1016/j.matcom.2007.06.007.

[61] M. H. Chen and Q. M. Shao. (1999). "Monte Carlo Estimation of Bayesian Credible and HPD Intervals". *Journal of Computational and Graphical Statistics.* **8** : 69-92. 10.2307/1390921.

[62] W. S. Abu El Azm, R. Aldallal, H. M. Aljohani, and S. G. Nassr. (2022). "Estimations of Competing Lifetime Data from Inverse Weibull Distribution Under Adaptive Progressively Hybrid Censoring". *Mathematical Biosciences and Engineering.* **19** (6): 6252-6276. 10.3934/mbe.2022292.

[63] A. Alrashidi, A. Rabie, A. A. Mahmoud, S. G. Nassr, M. S. A. Mustafa, A. Al Mutairi, E. Hussam, and M. M. Hossain. (2024). "Exponentiated Gamma Constant-Stress Partially Accelerated Life Tests with Unified Hybrid Censored Data: Statistical Inferences". *Alexandria Engineering Journal.* **88** : 268-275. 10.1016/j.aej.2023.12.066.

[64] S. G. Nassr, A. S. Hassan, E. M. Almetwally, A. Al Mutairi, R. H. Khashab, and N. M. Elharoun. (2023). "Statistical Inference of the Inverted Exponentiated Lomax Distribution Using Generalized Order Statistics with Application to COVID-19". *AIP Advances.* **13** (10): 105118-1-105118-15. 10.1063/5.0174540.

[65] S. G. Nassr, O. E. Abo-Kasem, R. H. Khashab, E. Alshawarbeh, S. S. Alshqaq, and N. M. Elharoun. (2025). "Reliability Analysis of Inverted Exponentiated Rayleigh Parameters via Progressive Hybrid Censoring Data with Applications in Medical Data". *PLoS One.* **20** (12): e0336169. 10.1371/journal.pone.0336169.

[66] S. G. Nassr, T. S. Taher, T. Alballa, and N. M. Elharoun. (2025). "Reliability Analysis of the Lindley Distribution via Unified Hybrid Censoring with Applications in Medical Survival and Biological Lifetime Data". *AIMS Mathematics.* **10** (6): 14943-14974. 10.3934/math.2025670.

[67] S. Ahmed, A. Al Mutairi, S. G. Nassr, H. Alsuhabi, M. Kamal, and M. U. Rehman. (2023). "A New Approach for Estimating Variance of a Population Employing Information Obtained from a Stratified Random Sampling". *Heliyon.* **9** (11): e21477. 10.1016/j.heliyon.2023.e21477.

[68] A. Al Mutairi, A. Alrashidi, N. T. Al-Sayed, M. Elgarhy, S. M. Behairy, and S. G. Nassr. (2023). "Bayesian and E-Bayesian Estimation Based on Constant Stress-Partially Accelerated Life Testing for Inverted Topp-Leone Distribution". *Open Physics.* **21** (1): 1-18. 10.1515/phys-2023-0126.

[69] M. M. Yousef, R. Alsultan, and S. G. Nassr. (2023). "Parametric Inference on Partially Accelerated Life Testing for the Inversed Kumaraswamy Distribution Based on Type-II Progressive Censoring Data". *Mathematical Biosciences and Engineering.* **20** (2): 1674-1694. 10.3934/mbe.2023076.

[70] S. G. Nassr and N. M. Elharoun. (2019). "Inference for Exponentiated Weibull Distribution Under Constant Stress Partially Accelerated Life Tests with Multiple Censored". *Communications for Statistical Applications and Methods*. **26** (2): 131-148. 10.29220/CSAM.2019.26.2.131.