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Research Article

Fixed Point Theorems for Semigroups of Lipschitzian Mappings

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Abstract
The main purpose of this paper is to extend the results of fixed point theorems for lipschitzian semigroups. The proofs we give follow the results of Ishihara, Suantai and Puengrattana theorems. Using one of the proofs, we also develop a fixed point theorem result for nonempty asymptotically total mapping semigroups on uniformly convex Banach spaces.

Keywords: semitopological semigroup; lipschitzian type semigroup; fixed point

1. INTRODUCTION

Let S be a semitopological semigroup, i.e., S is a semigroup with a Hausdorff topology such that for each $a \in S$ the mappings $s \to a \cdot s$ and $s \to s \cdot a$ from S to S are continuous. Let U be a nonempty subset of Banach space E. Then, family $\mathcal{S} = \{T_s : s \in S\}$ of mappings from U into itself is said to be a lipschitzian type semigroup on U if $\mathcal{S}$ satisfies the following:

a. $T_{st}(x) = T_sT_t(x)$ for all $s, t \in S$ and $x \in U$, **(1)**
b. the mapping $(s,x) \to T_s(x)$ from $S \times U$ into U is continuous when $S \times U$ has the product topology,
c. T_s is continuous for all $s \in S$, and **(2)**
d. there exists positive net $\{k_s\}$ such that for each $x \in U$ we have $\lim\sup_s c_s(x) = 0$, where

$$c_s(x) = \max\left\{\sup_{y \in U}(\|T_s(x) - T_s(y)\| - k_s\|x - y\|), 0\right\}$$

The class of lipschitzian type semigroups was introduced by Dung and Tan [1]. If $k_s = k$ for all $s \in S$, then a lipschitzian type semigroup reduces to a uniformly k-lipschitzian type semigroup [2]-[5]. Particularly, if $k_s = 1$ for all $s \in S$, thus, a lipschitzian type semigroup reduces to an asymptotically nonexpansive type semigroup. It is easy to see that the class of lipschitzian type semigroups contains the class of lipschitzian semigroups [6]-[8].

A semitopological semigroup S is left reversible if any two closed right ideals of S have nonvoid intersection. In this case, $(S, \succcurlyeq)$ is a directed system when the binary relation " $\succcurlyeq$ " on S is defined by $b \succcurlyeq a$ if and only if $\{a\} \cup \overline{aS} \supseteq \{b\} \cup \overline{bS}$. Dhompongsa et al. [9], Downing and Ray [10], and Ishihara and Takahashi [11] proved that in a Hilbert space a uniformly k-lipschitzian semigroup with $k < \sqrt{2}$ has a common fixed point. Later, Ishihara [11][12] generalized a result by proving that a lipschitzian semigroup in a Hilbert space has a common fixed point if $\lim\sup_s k_s < \sqrt{2}$. Casini and Maluta [13] and Ishihara and Takahashi [14] proved that a uniformly k-lipschitzain semigroup in a Banach space has a common fixed point if $k < \tilde{N}(E)^{-1/2}$, where $\tilde{N}(E)$ is the constant of uniformity of normal structure. Again, Ishihara [14] generalized a result by proving that a lipschitzian semigroup in a Banach space has a common fixed point if $\lim\sup_s k_s < \tilde{N}(E)^{-1/2}$. In these results, except [12], domain U of semigroups were assumed to be closed and convex [15]-[17].

Let S be a left reversible semitopological semigroup and U be a nonempty subset of Banach space E. Following Suantai and Puengrattana [18], family $\mathcal{S} = \{T_s : s \in S\}$ of mappings from U into itself is said to be a total asymptotically nonexpansive semigroup on U if $\mathcal{S}$ satisfies (1) and (2).

a. for every $x \in U$, the mapping $s \to T_s x$ from S into U is continuous, and
b. there exists nonnegative real numbers k_s, μ_s with $\lim_s k_s = 0$, $\lim_s \mu_s = 0$, and a strictly increasing continuous function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

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$\|T_s x - T_s y\| \leq \|x - y\| + k_s \phi(\|x - y\|) + \mu_s$
for all $x, y \in U$ and $s \in S$.

If $\phi(\lambda) = \lambda$, then a total asymptotically nonexpansive semigroup reduces to a generalized asymptotically nonexpansive semigroup. If $\phi(\lambda) = \lambda$ and $k_s = 0$ for all $s \in S$, then a total asymptotically nonexpansive semigroup reduces to an asymptotically nonexpansive semigroup. If $\phi(\lambda) = \lambda$ and $k_s = \mu_s = 0$ for all $s \in S$, then a total asymptotically nonexpansive semigroup reduces to a nonexpansive semigroup [19]-[22]. Suantai and Puengrattana proved that a total asymptotically nonexpansive semigroup in a uniformly convex Banach space has a common fixed point [18]. Again, in this result, the domain of the semigroup was assumed to be closed and convex.

2. MATERIALS AND METHODS

In this paper, we first show that if S is left reversible semitopological semigroup and if there exists a closed subset C of U such that $\cap_s \overline{co}\{T_t x : t \geqslant s\} \subseteq C$ for all $x \in U$ then a lipschitzian type semigroup on nonconvex domain in a Hilbert space with $\lim\sup_s k_s < \sqrt{2}$ has a common fixed point. Next, we prove that the theorem is valid in a Banach space E if $\lim\sup_s k_s < \tilde{N}(E)^{-1/2}$. These results extend the main results in Ishihara [12]. By a method of the proof of the theorem, we also prove that $\mathcal{S} = \{T_s : s \in S\}$ a total asymptotically nonexpansive semigroup on nonconvex domain in Banach space E still has a common fixed point. This result improves Theorem 3.3 [18].

3. RESULTS AND DISCUSSIONS

Let $\{B_\alpha\}_{\alpha\in\Lambda}$ be a decreasing net of bounded subsets of a Banach space E. For a nonempty subset C of E defined as

$$r(\{B_\alpha\}, x) = \inf_\alpha \sup\{\|x - y\| : y \in B_\alpha\}$$
$$r(\{B_\alpha\}, C) = \inf\{r(\{B_\alpha\}, x) : x \in C\}$$
$$\mathcal{A}(\{B_\alpha\}, C) = \{x \in C : r(\{B_\alpha\}, x) = r(\{B_\alpha\}, C)\}$$

We know that $r(\{B_\alpha\}, .)$ is a continuous convex function on E which satisfies the following:

$$|r(\{B_\alpha\}, x) - r(\{B_\alpha\}, y)| \leq \|x - y\| \leq r(\{B_\alpha\}, x) + r(\{B_\alpha\}, y)$$

for all $x, y \in E$. It is easy to see that if E is reflexive and if C is closed convex then $\mathcal{A}(\{B_\alpha\}, C)$ is nonempty. Moreover, if E is uniformly convex then it consists of a single point in which agreed with previous literature [23]. For a subset C, we denote by $\overline{co}C$ the closure of the convex hull of C, by $d(C)$ the diameter of C and by $R(C)$ the Chebyshev radius of C, i.e. $R(C) = \inf_{x\in C} \sup_{y\in C} \|x - y\|$. We define the uniformity $\tilde{N}(E)$ of normal structure E is the number.

$$\sup\left\{\frac{R(C)}{d(C)}\right\}$$

where the supremum is taken over all nonempty bounded convex $C \subseteq E$ with $d(C) > 0$. It is also known that if $\tilde{N}(E) < 1$, then E is reflexive, which is in line with previous literature [14].

The following lemmas play a crucial role in the proof of the theorems. We state the first lemma which was proved in the previous works[11][24] as:

Lemma 2.1. Let C be a nonempty closed convex subset of a Hilbert space H. Let $\{B_\alpha\}_{\alpha\in\Lambda}$ be a decreasing net of nonempty bounded subsets of H and let $\{a\} = \mathcal{A}(\{B_\alpha\}, C)$. Then

$$r(\{B_\alpha\}, C)^2 + \|a - x\|^2 \leq r(\{B_\alpha\}, x)^2$$

for all $x \in C$.

Lemma 2.2. Let C be a nonempty subset of a Hilbert space H [13]. Let $\{B_\alpha\}_{\alpha\in\Lambda}$ be a decreasing net of nonempty bounded subset of C. Then the asymptotic center a of $\{B_\alpha\}_{\alpha\in\Lambda}$ in C is an element of $\cap_\alpha \overline{co}B_\alpha$.

We also state the third lemma which was proved before [14] as:

Lemma 2.3. Let C be a closed convex subset of a reflexive Banach space E. Let $\{B_\alpha\}_{\alpha\in\Lambda}$ be a decreasing net of nonempty bounded closed convex subset of C and let $B = \cap_\alpha B_\alpha$. Then the following expression is obtained.

$$r(\{B_\alpha\}, B) \leq \tilde{N}(E) \inf_\alpha d(B_\alpha)$$

3.1. Fixed Point Theorems

We now prove a fixed point theorem for lipschitzian type semigroups defined on nonconvex domain in Hilbert spaces.

Theorem 3.1. Let U be a nonempty subset of a Hilbert space H and let S be a left reversible semitopological semigroup. Let $\mathcal{S} = \{T_s : s \in S\}$ be a lipschitzian type semigroup on U with $\lim_s \sup k_s < \sqrt{2}$. Suppose that $\{T_s y : s \in S\}$ is bounded for some $y \in U$ and there exists a closed subset C of U such that $\cap_s \overline{co}\{T_t x : t \succcurlyeq s\} \subseteq C$ for all $x \in U$. Then there exists a $z \in C$ such that $T_s z = z$ for all $s \in S$.

Proof. Let $B_s(x) = \{T_t x : t \succcurlyeq s\}$ for $s \in S$ and $x \in U$. Define $\{x_n : n \geq 0\}$ by induction as follows:

$$x_0 = y$$
$$\{x_{n+1}\} = \mathcal{A}(\{B_s(x_n)\}, \overline{co}U) \text{ for } n \geq 1$$

By Lemma 2.2, we have $x_n \in \overline{co} \cap_{s \in S} B_s(x_{n-1}) \subseteq C \subseteq U$ and hence $\{x_n\}$ is well defined. Let $r_n(x) = r(\{B_s(x_{n-1})\}, x)$ and $r_n = r(\{B_s(x_{n-1})\}, \overline{co}U)$ for $n \geq 1$. Then by Lemma 2.1, we have $\|x-u\|^2 \leq r_n(x)^2 - {r_n}^2$ for all $u \in \overline{co}U$ and $n \geq 1$. Putting $u = T_s x_n$, we have

$$\begin{aligned}\|x_n - T_s x_n\|^2 &\leq r_n(T_s x_n)^2 - {r_n}^2 \\ &= \left(\lim_t \sup\|T_t x_{n-1} - T_s x_n\|\right)^2 - {r_n}^2 \\ &\leq \left(\lim_t \sup\|T_s T_t x_{n-1} - T_s x_n\|\right)^2 - {r_n}^2 \\ &\leq \left(k_s \lim_t \sup\|T_t x_{n-1} - x_n\| + c_s(x_n)\right)^2 - {r_n}^2 \\ &= \left(k_s r_n + c_s(x_n)\right)^2 - {r_n}^2\end{aligned}$$

Let $\varsigma = \lim_s \sup {k_s}^2 - 1$. Then, we obtain

$$\begin{aligned}{r_{n+1}}^2 \leq r_{n+1}(x_n)^2 &= \lim_s \sup\|x_n - T_s x_n\|^2 \\ &\leq \lim_s \sup\left(k_s r_n + c_s(x_n)\right)^2 - {r_n}^2 \\ &\leq \left(r_n\left(\lim_s \sup k_s\right)\right)^2 - {r_n}^2 \\ &= \varsigma {r_n}^2 \leq \varsigma^n {r_1}^2\end{aligned}$$

for $n \geq 1$ all . Since

$$\|x_{n+1} - x_n\|^2 \leq 2\|x_{n+1} - T_t x_n\|^2 + 2\|T_t x_n - x_n\|^2$$

for all $t \in S$ and $n \geq 1$, we have

$$\begin{aligned}\|x_{n+1} - x_n\|^2 &\leq 2\lim_t \sup\|T_t x_n - x_{n+1}\|^2 + 2\lim_t \sup\|T_t x_n - x_n\|^2 \\ &\leq 2{r_{n+1}}^2 + 2r_{n+1}(x_n)^2 \leq 4ì^n {r_1}^2\end{aligned}$$

Therefore since $\varsigma < 1$, $\{x_n\}$ is a Cauchy sequence of C. Let $x_n \to z$. For each $s \in S$ we have

$$\|z - T_s z\|^2 \leq 2\|z - T_t x_n\|^2 + 2\|T_t x_n - T_s z\|^2$$

for all $t \in S$, hence

$$\begin{aligned}\|z - T_s z\|^2 &\leq 2\lim_t \sup\|z - T_t x_n\|^2 + 2\lim_t \sup\|T_t x_n - T_s z\|^2 \\ &\leq 2\lim_t \sup\|z - T_t x_n\|^2 + 2\lim_t \sup\|T_s T_t x_n - T_s z\|^2 \qquad \textbf{(3)}\end{aligned}$$

Note that

$$\begin{aligned}\lim_t \sup\|T_t x_n - z\|^2 &\leq 2\lim_t \sup\|T_t x_n - x_n\|^2 + 2\|x_n - z\|^2 \\ &= 2r_{n+1}(x_n)^2 + 2\|x_n - z\|^2 \\ &\leq 2ì^n {r_1}^2 + 2\|x_n - z\|^2 \to 0 \text{ as } n \to \infty\end{aligned}$$

From the continuity of T_s we also have

$$\lim_t \sup\|T_s T_t x_n - T_s z\|^2 \to 0 \text{ as } n \to \infty$$

Hence from (3) we have $T_s z = z$ for all $s \in S$. This completes the proof.

From Theorem 3.1 we capture the following result announced by Ishihara [12].

Corollary 3.2. Let U, H, and S as in Theorem 3.1 and let $\mathcal{S} = \{T_s : s \in S\}$ be a lipschitzian semigroup on U with $\lim_s \sup k_s < \sqrt{2}$. Suppose that $\{T_s y : s \in S\}$ is bounded for some $y \in U$ and there exists a closed subset C of U such that $\cap_s \overline{co}\{T_t x : t \succcurlyeq s\} \subseteq C$ for all $x \in U$. Then there exists a $z \in C$ such that $T_s z = z$ for all $s \in S$.

If we confine ourselves to an asymptotically nonexpansive type semigroup, we have the following result.

Teorema 3.3. Let U be a nonempty subset of a Hilbert space H and let S be a left reversible semitopological semigroup. Let $\mathcal{S} = \{T_s : s \in S\}$ be a lipschitzian type semigroup on U with $\lim_s \sup k_s \leq 1$. Suppose that $\{T_s x : s \in S\}$ is bounded and $\cap_s \overline{co}\{T_t x : t \succcurlyeq s\} \subseteq U$ for some $x \in U$. Then there exists a $z \in C$ such that $T_s z = z$ for all $s \in S$.

Proof. Let $B_s = \{T_t x : t \succcurlyeq s\}$ for $s \in S$ and let a be the asymptotic center of $\{B_s\}$ in $\overline{co}U$. Then by Lemma 2.1 and 2.2, we have

$$\begin{aligned}r(\{B_s\}, \overline{co}U)^2 + \|a - T_t a\|^2 &\leq r(\{B_s\}, T_t a)^2 \\ &= \left(\lim_s \sup\|T_s x - T_t a\|\right)^2 \\ &\leq \left(\lim_s \sup\|T_t T_s x - T_t a\|\right)^2 \\ &\leq \left(k_t r(\{B_s\}, a) + c_t(a)\right)^2\end{aligned}$$

for all $t \in S$. Hence, we have

$$\begin{aligned}\lim_t \sup\|T_t a - a\|^2 &\leq \lim_t \sup\left(k_t r(\{B_s\}, a) + c_t(a)\right)^2 - r(\{B_s\}, \overline{co}U)^2 \\ &\leq \left(r(\{B_s\}, a)\left(\lim_t \sup k_t\right)\right)^2 - r(\{B_s\}, \overline{co}U)^2 \\ &= \left(\left(\lim_t \sup k_t\right)^2 - 1\right) r(\{B_s\}, \overline{co}U)^2 \leq 0\end{aligned}$$

For each $s \in S$ we have

$$\|a - T_s a\|^2 \leq 2\|a - T_t a\|^2 + 2\|T_t a - T_s a\|^2$$

for all $t \in S$, hence

$$\begin{aligned}\|a - T_s a\|^2 &\le 2 \limsup_t \|T_t a - a\|^2 + 2 \limsup_t \|T_s T_t a - T_s a\|^2 \\ &= 2 \limsup_t \|T_s T_t a - T_s a\|^2\end{aligned}$$

Therefore, by the continuity of T_s we have $T_s a = a$ for all $s \in S$. This completes the proof.

From Theorem 3.3 we capture the following result announced by Ishihara [12].

Corollary 3.4. Let U, H, and S as in Theorem 3.3 and $\mathcal{S} = \{T_s : s \in S\}$ be a lipschitzian semigroup on U with $\limsup_s k_s \le 1$. Suppose that $\{T_s x : s \in S\}$ is bounded and $\cap_s \overline{co}\{T_t x : t \succcurlyeq s\} \subseteq U$ for some $x \in U$. Then there exists a $z \in C$ such that $T_s z = z$ for all $s \in S$.

Next, by a method similar to that of the proof of Theorem 3.1, we prove a fixed point theorem for lipschitzian type semigroups defined on nonconvex domain in Banach spaces.

Theorem 3.5. Let U be a nonempty subset of a Banach space E with $\tilde{N}(E) < 1$ and let S be a left reversible semitopological semigroup. Let $\mathcal{S} = \{T_s : s \in S\}$ be a lipschitzian type semigroup on U with $\limsup_s k_s < \tilde{N}(E)^{-1/2}$. Suppose that $\{T_s y : s \in S\}$ is bounded for some $y \in U$ and there exists a closed subset C of U such that $\cap_s \overline{co}\{T_t x : t \succcurlyeq s\} \subseteq C$ for all $x \in U$. Then there exists a $z \in C$ such that $T_s z = z$ for all $s \in S$.

Proof. Let $B_s(x) = \overline{co}\{T_t x : t \succcurlyeq s\}$ and let $B(x) = \cap_s B_s(x)$ for $s \in S$ and $x \in U$. Define $\{x_n : n \ge 0\}$ by induction as follows:

$$x_0 = y,$$
$$x_n \in \mathcal{A}\big(\{B_s(x_{n-1})\}, B(x_{n-1})\big) \text{ for } n \ge 1.$$

Well-definedness of $\{x_n\}$ follows from that $B(x) \subseteq C \subseteq U$ for all $x \in U$. Let $r_n(x) = r(\{B_s(x_{n-1})\}, x)$ and $r_n = r(\{B_s(x_{n-1})\}, B(x_{n-1}))$ for $n \ge 1$. Then from $x_n \in B(x_{n-1}) = \cap_t B_t(x_{n-1})$ for $n \ge 1$, and Lemma 2.3 we have

$$\begin{aligned}r_{n+1}(x_n) &= \limsup_s \|T_s x_n - x_n\| \le \limsup_s \limsup_t \|T_t x_{n-1} - T_s x_n\| \\ &\le \limsup_s \limsup_t \|T_s T_t x_{n-1} - T_s x_n\| \\ &\le \limsup_s \limsup_t \big(k_s \|T_t x_{n-1} - x_n\| + c_s(x_n)\big) \\ &\le \Big(\limsup_s k_s\Big) r_n(x_n) + \limsup_s c_s(x_n) \\ &= \Big(\limsup_s k_s\Big) r_n \le \Big(\limsup_s k_s\Big) \tilde{N}(E) \inf_s d\big(B_s(x_{n-1})\big)\end{aligned}$$

and

$$\begin{aligned}\inf_s d\big(B_s(x_{n-1})\big) &= \inf_s \sup\{\|T_a x_{n-1} - T_b x_{n-1}\| : a, b \succcurlyeq s\} \\ &\le \limsup_s \limsup_t \|T_t x_{n-1} - T_s x_{n-1}\| \\ &\le \limsup_s \limsup_t \|T_s T_t x_{n-1} - T_s x_{n-1}\| \\ &\le \limsup_s \limsup_t \big(k_s \|T_t x_{n-1} - x_{n-1}\| + c_s(x_{n-1})\big) \\ &\le \limsup_s \limsup_t \big(k_s \|T_t x_{n-1} - x_{n-1}\| + c_s(x_{n-1})\big) \\ &= \Big(\limsup_s k_s\Big) r_n(x_{n-1})\end{aligned}$$

Let $\varsigma = \Big(\limsup_s {k_s}^2\Big) \tilde{N}(E)$. Then we have

$$\begin{aligned}r_{n+1}(x_n) &\le \Big(\limsup_s k_s\Big) r_n \le \Big(\limsup_s {k_s}^2\Big) \tilde{N}(E) r_n(x_{n-1}) \\ &= \varsigma r_n(x_{n-1}) \le \varsigma^n r_1(x_0)\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1} - x_n\| &\le r\big(\{B_s(x_n)\}, B(x_n)\big) + r\big(\{B_s(x_n)\}, x_n\big) \\ &= r_{n+1} + r_{n+1}(x_n) \\ &\le \Big(\limsup_s k_s\Big) \tilde{N}(E) r_{n+1}(x_n) + r_{n+1}(x_n) \\ &\le \Big(\tilde{N}(E)\Big(\limsup_s k_s\Big) + 1\Big) \text{ì}^n r_1(x_0)\end{aligned}$$

for all $n \ge 1$. So, $\{x_n\}$ is a Cauchy sequence of C and hence $\{x_n\}$ converges to a point $z \in C$. For each $s \in S$ we have

$$\|z - T_s z\| \le \|z - T_t x_n\| + \|T_t x_n - T_s z\|$$

for all $t \in S$, hence

$$\begin{aligned}\|z - T_s z\| &\le \limsup_t \|z - T_t x_n\| + \limsup_t \|T_t x_n - T_s z\| \\ &\le \limsup_t \|z - T_t x_n\| + \limsup_t \|T_s T_t x_n - T_s z\| \qquad \textbf{(4)}\end{aligned}$$

Note that

$$\begin{aligned}\limsup_t \|T_t x_n - z\| &\le \limsup_t \|T_t x_n - x_n\| + \|x_n - z\| \\ &= r_{n+1}(x_n) + \|x_n - z\| \\ &\le \varsigma^n r_1(x_0) + \|x_n - z\| \to 0 \quad \text{as} \quad n \to \infty\end{aligned}$$

From the continuity of T_s we also have

$$\limsup_t \|T_s T_t x_n - T_s z\| \to 0 \quad \text{as} \quad n \to \infty$$

Hence from (4) we have $T_s z = z$ for all $s \in S$. This completes the proof.

From Theorem 3.5 we also capture the following result announced by Ishihara [12].

Corollary 3.6. Let U, E, and S as in Theorem 3.5 and let $\mathcal{S} = \{T_s : s \in S\}$ be a lipschitzian semigroup on U with $\limsup_s k_s < \tilde{N}(E)^{-1/2}$. Suppose that $\{T_s y : s \in S\}$ is bounded for some $y \in U$ and there exists a closed subset C of U such that $\cap_s \overline{co}\{T_t x : t \succcurlyeq s\} \subseteq C$ for all $x \in U$. Then there exists a $z \in C$ such that $T_s z = z$ for all $s \in S$.

Now we state a fixed point theorem for total asymptotically nonexpansive semigroups defined on nonconvex domain in Banach spaces. We use a method of the proof of Theorem 3.5 to prove the following result.

Theorem 3.7. Let U be a nonempty subset of a Banach space E with $\tilde{N}(E) < 1$ and let S be a left reversible semitopological semigroup. Let $\mathcal{S} = \{T_s : s \in S\}$ be a total asymptotically nonexpansive semigroup on U. Suppose that $\{T_s y : s \in S\}$ is bounded for some $y \in U$ and there exists a closed subset C of U such that $\cap_s \overline{co}\{T_t x : t \geqslant s\} \subseteq C$ for all $x \in U$. Then there exists a $z \in C$ such that $T_s z = z$ for all $s \in S$.

Proof. Fix $\varsigma \in (\tilde{N}(E), 1)$. Let $B_s(x) = \overline{co}\{T_t x : t \geqslant s\}$ and let $B(x) = \cap_s B_s(x)$ for $s \in S$ and $x \in U$. Define $\{x_n : n \geq 0\}$ by induction as follows:

$$x_0 = y$$
$$x_n \in \mathcal{A}(\{B_s(x_{n-1})\}, B(x_{n-1})) \text{ untuk } n \geq 1$$

Well-definedness of $\{x_n\}$ follow from that $B(x) \subseteq C \subseteq U$ for all $x \in U$. Let $r_n(x) = r(\{B_s(x_{n-1})\}, x)$ and $r_n = r(\{B_s(x_{n-1})\}, B(x_{n-1}))$ for $n \geq 1$. Then from $x_n \in B(x_{n-1}) = \cap_t B_t(x_{n-1})$ for $n \geq 1$, and Lemma 2.3 we have

$$\begin{aligned} r_{n+1}(x_n) &= \lim_s \sup \|T_s x_n - x_n\| \leq \lim_s \sup \lim_t \sup \|T_t x_{n-1} - T_s x_n\| \\ &\leq \lim_s \sup \lim_t \sup \|T_s T_t x_{n-1} - T_s x_n\| \\ &\leq \lim_s \sup \lim_t \sup (\|T_t x_{n-1} - x_n\| + k_s \phi(\|T_t x_{n-1} - x_n\|) + \grave{\imath}_s) \\ &\leq r_n(x_n) + \left(\lim_s \sup \left(k_s \lim_t \sup \phi(\|T_t x_{n-1} - x_n\|) + \grave{\imath}_s\right)\right) \\ &= r_n \leq \varsigma \inf_s d(B_s(x_{n-1})) \end{aligned}$$

and

$$\begin{aligned} \inf_s d(B_s(x_{n-1})) &\leq \lim_s \sup \lim_t \sup \|T_t x_{n-1} - T_s x_{n-1}\| \\ &\leq \lim_s \sup \lim_t \sup \|T_s T_t x_{n-1} - T_s x_{n-1}\| \\ &\leq \lim_s \sup \lim_t \sup (\|T_t x_{n-1} - x_{n-1}\| + k_s \phi \\ &\quad (\|T_t x_{n-1} - x_n\|) + \grave{\imath}_s) \\ &\leq r_n(x_{n-1}) + \left(\lim_s \sup \left(k_s \lim_t \sup \phi(\|T_t x_{n-1} \right.\right. \\ &\quad \left.\left. - x_n\|) + \grave{\imath}_s\right)\right) \\ &= r_n(x_{n-1}) \end{aligned}$$

Then we have

$$r_{n+1}(x_n) \leq \varsigma r_n(x_{n-1}) \leq \varsigma^n r_1(x_0)$$

and

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq r(\{B_s(x_n)\}, B(x_n)) + r(\{B_s(x_n)\}, x_n) \\ &= r_{n+1} + r_{n+1}(x_n) \\ &\leq \varsigma r_{n+1}(x_n) + r_{n+1}(x_n) \\ &\leq 2\varsigma^n r_1(x_0) \end{aligned}$$

for all $n \geq 1$. Then as in the proof of Theorem 3.5, it follows that the sequence $\{x_n\}$ convergences to some $z \in C$ for which $T_s z = z$ for all $s \in S$. This completes the proof.

From Theorem 3.7 we are ready to capture the following result announced by Suantai and Puengrattana, who also give an alternative proof [18].

Corollary 3.8. Let S be a left reversible semitopological semigroup, U be a nonempty closed convex subset of a uniformly convex Banach space E, and $\mathcal{S} = \{T_s : s \in S\}$ be a total asymptotically nonexpansive semigroup on U. Then $\mathcal{S}$ has a common fixed point if and only if $\{T_s x : s \in S\}$ is bounded for some $x \in U$.

Proof. The necessity is obvious. For sufficiency, this follows since a uniformly convex Banach space E has a property $\tilde{N}(E) < 1$ [25]. These complete the proof.

4. CONCLUSIONS

We conclude the paper by stating the Hilbert space version of Theorem 3.7. The proof is too similar to that of Theorem 3.3 and is therefore omitted.

Theorem 3.9. Let U be a nonempty subset of a Hilbert space H and let S be a left reversible semitopological semigroup. Let $\mathcal{S} = \{T_s : s \in S\}$ be a total asymptotically nonexpansive semigroup on U. Suppose that $\{T_s x : s \in S\}$ is bounded and $\cap_s \overline{co}\{T_t x : t \geqslant s\} \subseteq U$ for some $x \in U$. Then there exists a $z \in C$ such that $T_s z = z$ for all $s \in S$.

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Conflicts of Interest

The author(s) declare no conflict of interest.

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